Electromagnetic Continuum Gyrokinetic Turbulence Simulations in the Tokamak Edge

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Abstract

Gkeyll, a full-F continuum gyrokinetic code, is being developed to study turbulence in the edge region of fusion devices. The edge region involves large-amplitude fluctuations, electromagnetic effects, and plasma interactions with material walls, making it more computationally challenging than the core region. Gkeyll models the turbulence by solving the 5-D full-F gyrokinetic system in Hamiltonian form using an energy conserving high-order discontinuous Galerkin scheme. The code has been extended to include self-consistent electromagnetic perturbations using a symplectic ($v_{||}$) formulation. We present some linear benchmarks that illustrate the success of the electromagnetic scheme and the avoidance of the Ampere cancellation problem. We also present nonlinear electromagnetic turbulence simulations in a model SOL geometry with sheath boundary conditions on open field lines. The effects of electromagnetic fluctuations on the turbulence are discussed.

Status of gyrokinetics in `Gkeyll`

- Pioneering work by Eric Shi[1] led to 5D electrostatic full-F GK simulations of LAPD and NSTX-like helical SOL with sheath BCs
- Discontinuous Galerkin (DG) discretization scheme
  - high order method, local and parallelizable
  - conserves energy for Hamiltonian systems (like GK)
- We have been developing a new version of `Gkeyll`
  - Moving from nodal to modal DG representation $\rightarrow$ orthonormal basis functions, quadrature-free, computer algebra-generated solver kernels (much easier to generalize to higher dimensionality/polynomial order), $\mathcal{O}(10)$ faster
  - Much simpler user interface, details abstracted away
- Have reproduced many of Shi's results with new version of `Gkeyll`
- **New nonlinear SOL simulations with electromagnetics**

[1] See 2017 thesis; JPP 2017 paper on LAPD; and PoP 2019 paper on Helical SOL

What about electromagnetics?

- Electromagnetic effects are especially important in the edge and SOL, where steep gradients can push the plasma close to the ideal-MHD stability threshold and produce stronger turbulence
- Including electromagnetic fluctuations has proved challenging in some PIC codes, in part due to the well-known Ampère cancellation problem
- Continuum gyrokinetic codes for core turbulence have avoided the Ampère cancellation issue
- As `Gkeyll` uses a continuum formulation, we expect that we can handle electromagnetic effects in the edge and SOL in a stable and efficient manner

Hamiltonian ($p_\parallel$) vs. Symplectic ($v_\parallel$) formulation of EMGK

In the Hamiltonian gyrokinetic formalism (see e.g. Brizard & Hahm, 2007), there are two formulations for including electromagnetic fluctuations:

- Hamiltonian formulation: $p_\parallel = mv_\parallel + qA_\parallel$

$$\frac{\partial f}{\partial t} = \{H, f\}$$

$$H = \frac{1}{2m}p_\parallel^2 + \mu B + q\phi = \frac{1}{2m}(mv_\parallel + qA_\parallel)^2 + \mu B + q\phi \qquad \mathbf{B}^* = \mathbf{B_0} + \frac{1}{q}p_\parallel \nabla \times \hat{\mathrm{b}}$$

- Symplectic formulation: $p_\parallel = mv_\parallel$

$$\frac{\partial f}{\partial t} = \{H, f\} + \frac{q}{m}\frac{\partial f}{\partial v_\parallel}\frac{\partial A_\parallel}{\partial t}$$

$$H = \frac{1}{2}mv_\parallel^2 + \mu B + q\phi \qquad \mathbf{B}^* = \mathbf{B_0} + \frac{m}{q}v_\parallel \nabla \times \hat{\mathrm{b}} + \delta\mathbf{B}_\perp$$

Poisson bracket:

$$\{F, G\} = \frac{\mathbf{B}^*}{B_\parallel^*} \cdot \left(\nabla F \frac{\partial G}{\partial p_\parallel} - \frac{\partial F}{\partial p_\parallel}\nabla G\right) - \frac{\hat{\mathrm{b}}}{qB_\parallel^*} \times \nabla F \cdot \nabla G$$

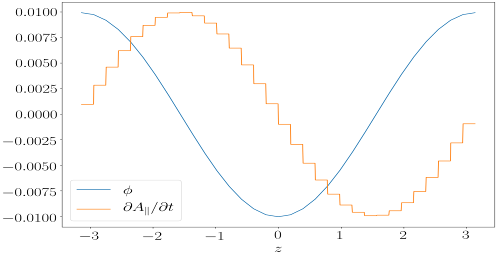
Hamiltonian ($p_\parallel$) vs. Symplectic ($v_\parallel$) formulation of EMGK

In `Gkeyll`'s DG scheme, the distribution function and other fields can be discontinuous across cell boundaries, but **energy is conserved only if the Hamiltonian is continuous**

- Hamiltonian ($p_\parallel$) $\Rightarrow$ both ϕ & $A_\parallel$ must be continuous
- Symplectic ($v_\parallel$) $\Rightarrow$ ϕ must be continuous, but $A_\parallel$ (and $\frac{\partial A_\parallel}{\partial t}$) can be discontinuous in parallel direction

Ex) MHD limit, system wants $E_\parallel = 0 \Rightarrow \frac{\partial \phi}{\partial z} = -\frac{\partial A_\parallel}{\partial t}$

Piecewise linear $\phi \Rightarrow$ piecewise constant $\frac{\partial \phi}{\partial z} \Rightarrow$ piecewise constant $\frac{\partial A_\parallel}{\partial t}$



Ampère cancellation problem: Hamiltonian formulation

In Hamiltonian formulation, Ampère's law becomes

$$\left(-\nabla_\perp^2 + C_n \sum_s \frac{\mu_0 q}{m} \int d^3p\ f\right) A_\parallel = C_j\ \mu_0 \sum_s \frac{q}{m^2} \int d^3p\ p_\parallel f$$

The "cancellation problem" arises when there are small errors in the calculation of the integrals. These errors are represented by C_n and C_j (which should both be exactly 1 in the exact system).

The simplest Alfvén wave dispersion relation (slab geometry, uniform Maxwellian background with stationary ions) becomes (with $\hat{\beta} \equiv \frac{\beta_e}{2}\frac{m_i}{m_e}$)

$$\omega^2 = \frac{k_\parallel^2 v_A^2}{C_n + k_\perp^2 \rho_s^2/\hat{\beta}} \left[1 + (C_n - C_j)\frac{\hat{\beta}}{k_\perp^2 \rho_s^2}\right]$$

This reduces to the correct result if integrals calculated consistently, so that $C_n = C_j$, but if not there will be large errors for modes with $\hat{\beta}/k_\perp^2 \rho_s^2 \gg 1$.

Ampère cancellation problem: symplectic formulation

In symplectic formulation, Ampère's law is

$$-\nabla_\perp^2 A_\parallel = \mu_0 \sum_s q \int d^3v \; v_\parallel f$$

However, we need a way to handle the $\frac{\partial A_\parallel}{\partial t}$ term that appears in the GK equation. One way is to take $\frac{\partial}{\partial t}$ of Ampère's law, which gives an Ohm's law:

$$-\nabla_\perp^2 \frac{\partial A_\parallel}{\partial t} = \mu_0 \sum_s q \int d^3v \; v_\parallel \frac{\partial f}{\partial t} = \mu_0 \sum_s q \int d^3v \; v_\parallel \left[\{H, f\} + \frac{q}{m} \frac{\partial f}{\partial v_\parallel} \frac{\partial A_\parallel}{\partial t} \right]$$

$$\Rightarrow \left(-\nabla_\perp^2 + C_n \sum_s \frac{\mu_0 q^2}{m} \int d^3v \; f \right) \frac{\partial A_\parallel}{\partial t} = C_j \; \mu_0 \sum_s q \int d^3v \; v_\parallel \{H, f\}$$

Same dispersion relation, but integrals over $v_\parallel$, not $p_\parallel$. These can easily be calculated consistently so that $C_n = C_j$ and there is no cancellation problem.

We choose symplectic formulation of EMGK

Electromagnetic GK equation:

$$\frac{\partial f}{\partial t} = \{H, f\} + \frac{q}{m}\frac{\partial f}{\partial v_\parallel}\frac{\partial A_\parallel}{\partial t} + C[f] + S \qquad (1)$$
$$= \frac{\partial f^\star}{\partial t} + \frac{q}{m}\frac{\partial f}{\partial v_\parallel}\frac{\partial A_\parallel}{\partial t},$$

with $H = \frac{1}{2}mv_\parallel^2 + \mu B + q\phi$, and $\frac{\partial f^\star}{\partial t} \equiv \{H, f\} + C[f] + S$

Quasineutrality equation (long-wavelength):

$$-\nabla \cdot \sum_s \frac{mn_0}{B^2}\nabla_\perp \phi = \sum_s q \int d^3v\ f \qquad (2)$$

Ohm's law: *solve directly for* $\partial A_\parallel / \partial t$

$$\left(-\nabla_\perp^2 + \sum_s \frac{\mu_0 q^2}{m}\int d^3v\ f\right)\frac{\partial A_\parallel}{\partial t} = \mu_0 \sum_s q \int d^3v\ v_\parallel \frac{\partial f^\star}{\partial t} \qquad (3)$$

Parallel Ampère equation: *only used for initial condition on* $A_\parallel$

$$-\nabla_\perp^2 A_\parallel = \mu_0 \sum_s q \int d^3v\ v_\parallel f \qquad (4)$$

Explicit time-advance scheme

We use a multi-stage SSP-RK method. Can be built from multiple forward Euler steps.
Forward Euler scheme: Given f^n and $A_\parallel^n$ at the beginning of timestep n,

1. Calculate ϕ^n:

$$-\nabla \cdot \sum_s \frac{mn_0}{B^2} \nabla_\perp \phi^n = \sum_s q \int d^3v \ f^n$$

2. Calculate partial GK RHS:

$$\left(\frac{\partial f^\star}{\partial t}\right)^n = \{H^n, f^n\}^n + C[f^n] + S^n$$

3. Calculate $\left(\frac{\partial A_\parallel}{\partial t}\right)^n$:

$$\left(-\nabla_\perp^2 + \sum_s \frac{\mu_0 q^2}{m} \int d^3v \ f^n\right)\left(\frac{\partial A_\parallel}{\partial t}\right)^n = \mu_0 \sum_s q \int d^3v \ v_\parallel \left(\frac{\partial f^\star}{\partial t}\right)^n$$

4. Advance f^{n+1} and $A_\parallel^{n+1}$:

$$f^{n+1} = f^n + \Delta t \left[\left(\frac{\partial f^\star}{\partial t}\right)^n + \frac{q}{m}\frac{\partial f^n}{\partial v_\parallel}\left(\frac{\partial A_\parallel}{\partial t}\right)^n\right], \qquad A_\parallel^{n+1} = A_\parallel^n + \Delta t \left(\frac{\partial A_\parallel}{\partial t}\right)^n$$

Linear Benchmark: Kinetic Alfvén Waves

In slab geometry, with a uniform Maxwellian background and stationary ions, the linearized GK equation reduces to

$$\frac{\partial f_e}{\partial t} + v_{\|} \frac{\partial f_e}{\partial z} = v_{\|} F_{Me} \left(\frac{\partial \phi}{\partial z} + \frac{\partial A_{\|}}{\partial t} \right).$$

Taking a single Fourier mode with perpendicular wavenumber $k_\perp$ and parallel wavenumber $k_{\|}$, the field equations become

$$k_\perp^2 \frac{m_i n_0}{B^2} \phi = -e \int dv_{\|} \ f_e$$

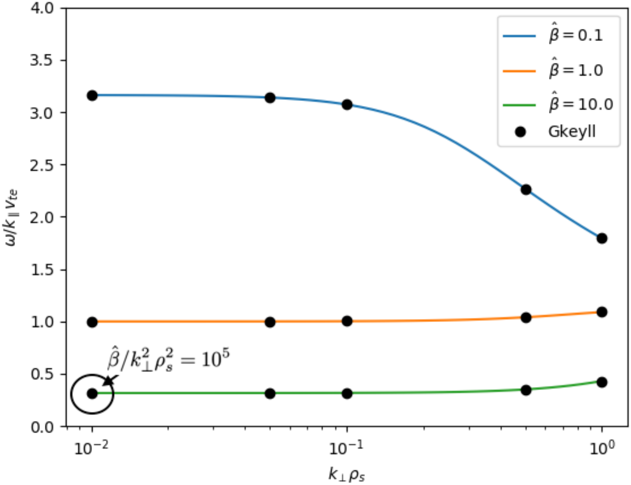
$$k_\perp^2 A_{\|} = -\mu_0 e \int dv_{\|} \ v_{\|} f_e.$$

The KAW dispersion relation is then

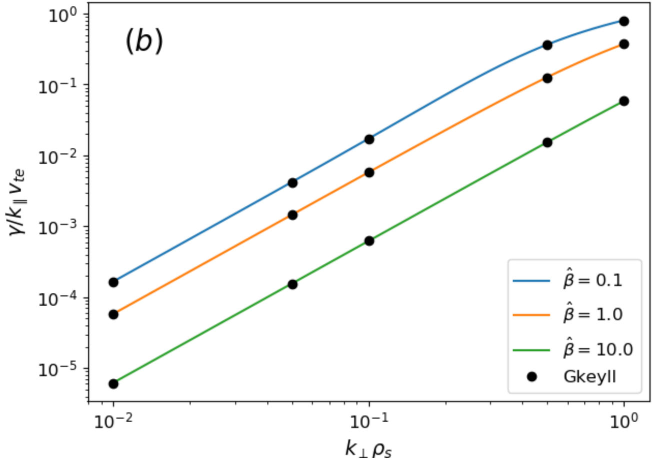
$$\omega^2 \left[1 + \frac{\omega}{\sqrt{2} k_{\|} v_{te}} Z\left(\frac{\omega}{\sqrt{2} k_{\|} v_{te}} \right) \right] = \frac{k_{\|}^2 v_{te}^2}{\hat{\beta}} \left[1 + k_\perp^2 \rho_s^2 + \frac{\omega}{\sqrt{2} k_{\|} v_{te}} Z\left(\frac{\omega}{\sqrt{2} k_{\|} v_{te}} \right) \right],$$

where $\hat{\beta} = (\beta_e/2) m_i/m_e$, and $Z(x)$ is the usual plasma dispersion function.

Linear Benchmark: Kinetic Alfvén Waves



Linear Benchmark: Kinetic Alfvén Waves



Linear Benchmark: Kinetic Ballooning Mode Instability

In local limit, KBM dispersion relation given by solving

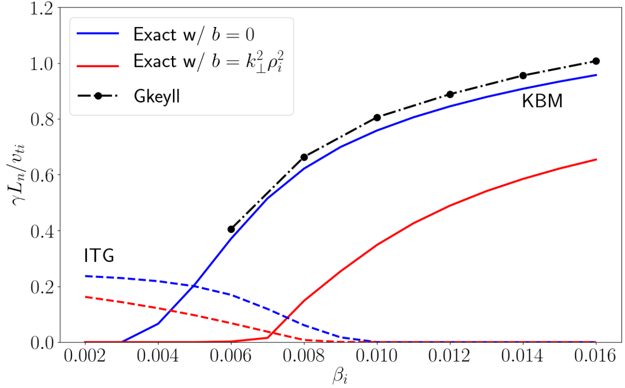
$$\omega\left[\tau + k_\perp^2 + \Gamma_0(b) - P_0\right]\phi = \left[\tau(\omega - \omega_{*e}) - k_\parallel P_1\right]\frac{\omega}{k_\parallel}A_\parallel$$

$$\frac{2k_\parallel^2 k_\perp^2}{\beta_i}A_\parallel = k_\parallel\left[k_\parallel P_1 - \tau(\omega - \omega_{*e})\right]\phi - \left[k_\parallel^2 P_2 - \tau\left(\omega(\omega - \omega_{*e} - 2\omega_d(\omega - \omega_{*e}(1+\eta_e)))\right)\right]A_\parallel$$

with

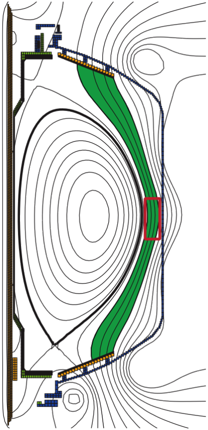
$$P_m = \int_0^\infty dv_\perp\ v_\perp \int_{-\infty}^\infty dv_\parallel\ \frac{1}{\sqrt{2\pi}}e^{-(v_\parallel^2+v_\perp^2)/2}(v_\parallel)^m\frac{\omega - \omega_{*i}\left[1+\eta_i(v^2/2 - 3/2)\right]}{\omega - k_\parallel v_\parallel - \omega_d(v_\parallel^2 + v_\perp^2/2)}J_0^2(v_\perp\sqrt{b})$$

Linear Benchmark: Kinetic Ballooning Mode Instability



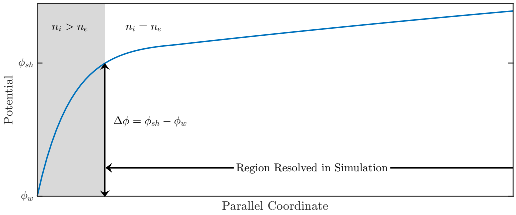
$k_\perp \rho_i = 0.5,\ k_\| L_n = 0.1,\ R/L_n = 5,\ R/L_{Ti} = 12.5,\ R/L_{Te} = 10,\ \tau = 1$

EM turbulence in NSTX-like helical SOL



- Simple helical model of tokamak SOL
  - Like the green region, but straightened out to vertical flux surfaces
  - Field-aligned simulation domain that follows field lines from bottom divertor plate, around the torus, to the top divertor plate
  - All bad curvature; brings in interchange instability drive
- Parameters taken from NSTX SOL measurements; Real deuterium mass ratio, Lenard-Bernstein collisions
- Conducting sheath boundary conditions at the divertor plates
- Radially-localized source around $x = 1.3$ cm models flux of particles and heat across separatrix from core
- See Shi *et al.*, 2019 (PoP)

Conducting-Sheath Boundary Conditions



- Need to model effects of non-neutral sheath using BCs
- Get ϕ_{sh}(x,y) from solving GK Poisson equation, then use $\Delta\phi = \phi_{sh} - \phi_w$ to reflect low-$v_{||}$ electrons entering sheath
  - Kinetic version of sheath BCs used in some fluid models (also similar to some gyrofluid sheath BCs)
- Potential self-consistently relaxes to ambipolar-parallel-outflow state
- Allows local currents into and out of the wall
- No BC applied at sheath to ions (free outflow)

Sheath-Model Boundary Conditions for Electrons

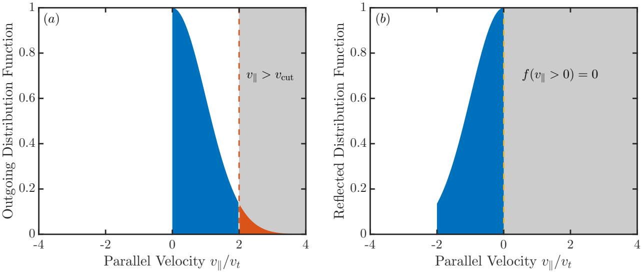


Figure: Illustration of sheath-model boundary condition. (a) Outgoing electrons with $v_{||} > v_{\rm cut} = \sqrt{2e\Delta\phi/m}$ are lost into the wall, where $\Delta\phi = \phi_{sh} - \phi_w$, ϕ_{sh} is determined from the GK Poisson equation, and $\phi_w = 0$ for a grounded wall. (b) The rest of the outgoing particles $(0 < v_{||} < v_{\rm cut})$ are reflected back into the plasma.

Source and parameters for NSTX-like helical SOL

Parameter	Value
ρ_{s0}	2.9 mm
ρ_e	0.048 mm
B_{axis}	0.5 T
B_v/B_z	0.3
L_v	2.4 m
L_z	8 m
L_x	$50\rho_s = 14.6$ cm
L_y	$100\rho_s = 29.1$ cm
n_0	7×10^{18} m^{-3}
$T_{i,\text{src}} = T_{e,\text{src}}$	74 eV
$T_{i,\text{sep}}$	40 eV
$T_{e,\text{sep}}$	25 eV
λ_{ee}	0.96 m
λ_{ii}	3.5 m
$c_s/\sqrt{R\lambda_p}$	1.9×10^5 s^{-1}

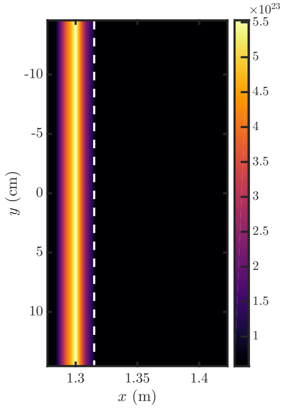
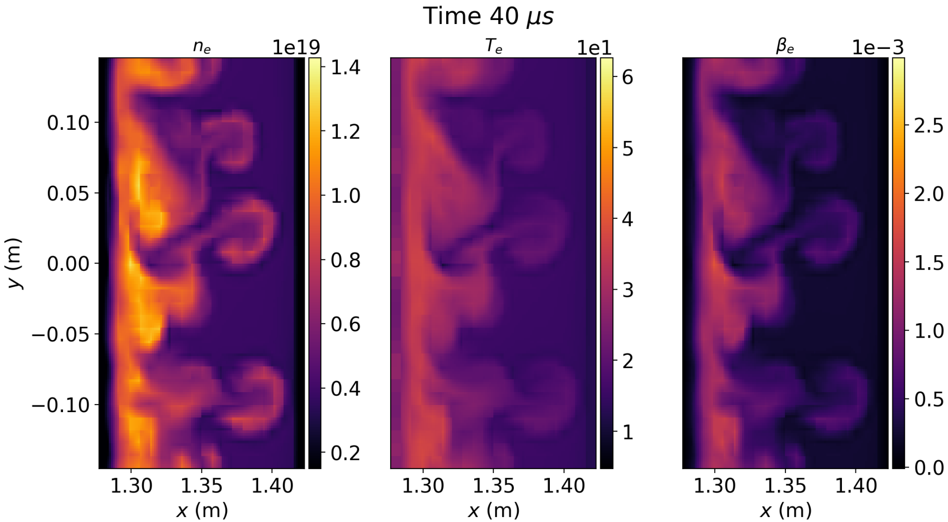
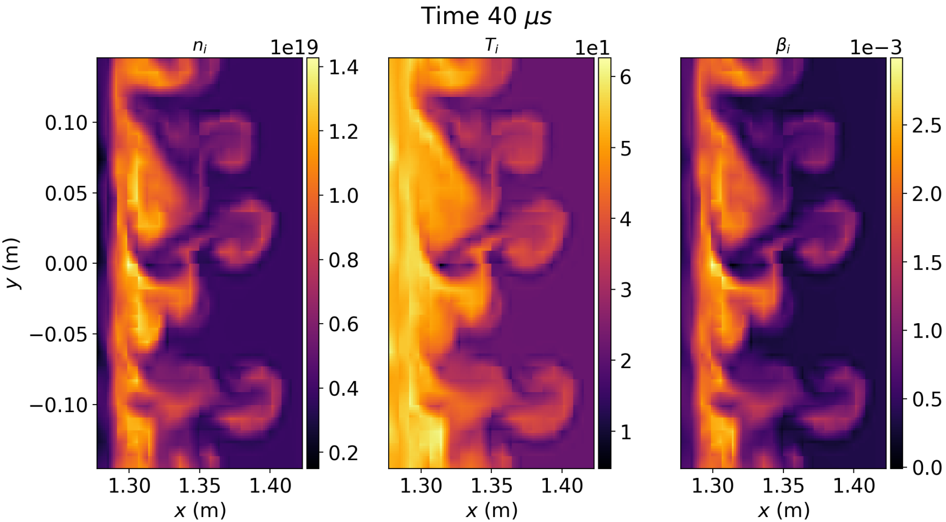


Figure: Midplane particle source for helical-SOL simulations in the perpendicular (x, y) plane.

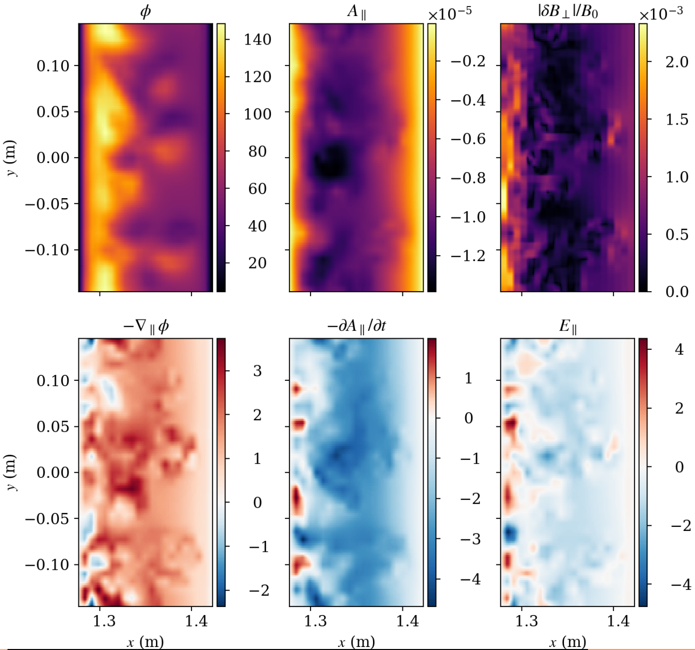
EM turbulence in NSTX-like helical SOL model



EM turbulence in NSTX-like helical SOL model



Time 40 μs


ϕ
$A_\parallel$
$\times 10^{-5}$
$|\delta B_\perp|/B_0$
$\times 10^{-3}$
$-\nabla_\parallel \phi$
$-\partial A_\parallel/\partial t$
$E_\parallel$
y (m)
x (m)
0.10
0.05
0.00
−0.05
−0.10
1.3
1.4
140
120
100
80
60
40
20
−0.2
−0.4
−0.6
−0.8
−1.0
−1.2
2.0
1.5
1.0
0.5
0.0
3
2
1
0
−1
−2
−3
−4
4
−4

Summary & Future Work

- We have a new version of the `Gkeyll` code that is faster and includes EM
- We have demonstrated that our formulation and scheme for EMGK is effective and avoids the Ampère cancellation problem
- We have successfully completed some basic linear EMGK benchmarks
- **We have performed preliminary nonlinear full-F continuum EMGK SOL simulations**
- In-progress/Future Work:
  - Detailed comparison of ES and EM GK simulations in helical SOL geometry
  - Generalize the geometry to better model NSTX SOL, and also to include closed field line regions
  - Include FLR effects (beyond the first order polarization drift)