# VORTICES AND SPECTRAL TRANSFER IN THE MODIFIED TERRY-HORTON EQUATION

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## Turbulence, Zonal Flows, and the Dimits Shift

- Ion temperature gradients can cause turbulence to develop in toroidal plasmas, driving heat transport.
- The temperature gradient needed for instability can be calculated using linear theory.

**The Dimits shift is the nonlinear upshift of the critical temperature gradient that signals the onset of turbulence.**

- This is caused by a shearing away of turbulent streamers by poloidal zonal flows (ZF) generated through a secondary instability.
- This has been witnessed in both gyrofluid and gyrokinetic simulations.
- If the linear drive is sufficiently large, the system undergoes a tertiary instability, and turbulence ensues.

## Questions and Motivation

- **Why should the Dimits shift exist?**
- **What is the true nature of the tertiary instability?**
- **Can the size of the shift be calculated?**

**In order to answer these questions, a simple model is needed that captures the Dimits shift with a minimal amount of physics.**

## The Modified Terry-Horton Equation

The Terry-Horton equation is modified to include proper adiabatic electron response and is made to capture Rosenbluth-Hinton states:

$$\frac{\partial \zeta}{\partial t} + \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}\zeta = \beta\frac{\partial \varphi}{\partial y} - \widehat{\alpha} D\zeta,$$

where $\boldsymbol{v} = (-\partial_y\varphi, \partial_x\varphi)$, $D$ is a damping operator and

$$\zeta \doteq (\nabla_\perp^2 + \mathrm{i}\widehat{\delta} - \widehat{\alpha})\varphi$$

is the generalized vorticity, where

$\nabla_\perp^2\varphi \rightarrow$ ion polarization,
$\mathrm{i}\widehat{\delta}\varphi \rightarrow$ non-adiabatic electron response,
$\widehat{\alpha}\varphi \rightarrow$ adiabatic electron response.

The modification is done by introducing the operator

$$\widehat{\alpha}\varphi \doteq \varphi - \langle\varphi\rangle \doteq \varphi - \frac{1}{L_y}\int_0^{L_y} \mathrm{d}y\,\varphi,$$

which is zero when acting on zonal modes and unity otherwise. The modified Terry Horton equation has the linear growth rates $\lambda_{\boldsymbol{k}} = \gamma_{\boldsymbol{k}} - \mathrm{i}\omega_{\boldsymbol{k}}$ given by

$$\gamma_{\boldsymbol{k}} = -D_{\boldsymbol{k}} + \frac{\beta k_y \widehat{\delta}_{\boldsymbol{k}}}{(1+k_\perp^2)^2 + \widehat{\delta}_{\boldsymbol{k}}^2}, \qquad \omega_{\boldsymbol{k}} = \frac{\beta k_y(1+k_\perp^2)}{(1+k_\perp^2)^2 + \widehat{\delta}_{\boldsymbol{k}}^2}.$$

This system contains destabilizing electron effects through $\mathrm{i}\widehat{\delta}\varphi$ and can be made to model ion Landau damping as well.

**Parameters used are $\widehat{\delta}_{\boldsymbol{k}} = 1.5k_y$, $D = 1 - 0.01\nabla_\perp^2$.**

## Previous Results

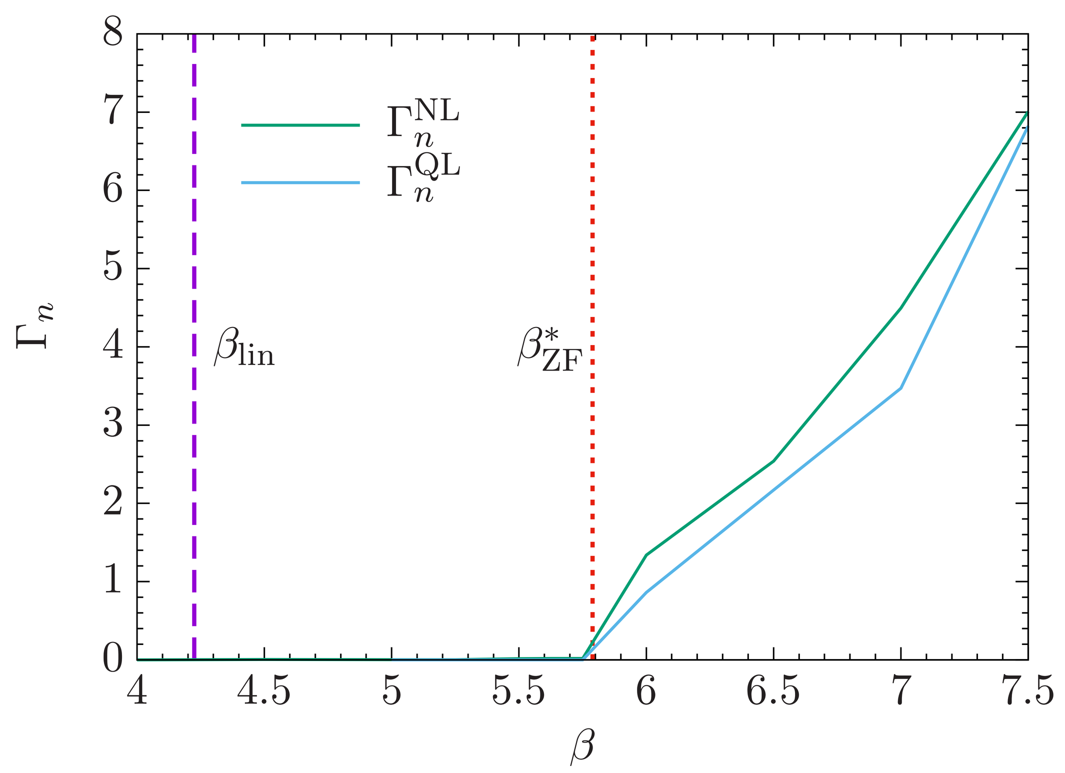


Figure 1: Radial particle flux $\Gamma_n$ for the nonlinear and quasilinear systems as a function of $\beta$. The dashed-dotted line denotes the linear threshold for instability $\beta_{\mathrm{lin}}$, while the dashed line denotes the calculated upshifted critical gradient, $\beta_{\mathrm{ZF}}^*$.

### Heuristic Calculation of the Dimits Shift

The Dimits shift can be estimated by calculating the range of the interaction ($q_x^*$) and the stability condition for the collection of nonlocally coupled modes (phase-space expansion):

$$\left.\frac{\partial\gamma(0,p_y)}{\partial p_y}\right|_{p_y=p_y^*} = 0, \qquad \textbf{Fastest Growing Mode},$$

$$\int_0^{q_x^*} \mathrm{d}k_x\,\gamma(k_x, p_y^*) = 0, \qquad \textbf{Phase-Space Expansion/ Stability},$$

$$\begin{Bmatrix}\Theta = 0\\ \Omega = 0\end{Bmatrix}, \qquad \textbf{Interaction Range in } k \textbf{ space.}$$

These equations are solved for $\beta_{\mathrm{ZF}}^*$, the value of the upshifted critical gradient.

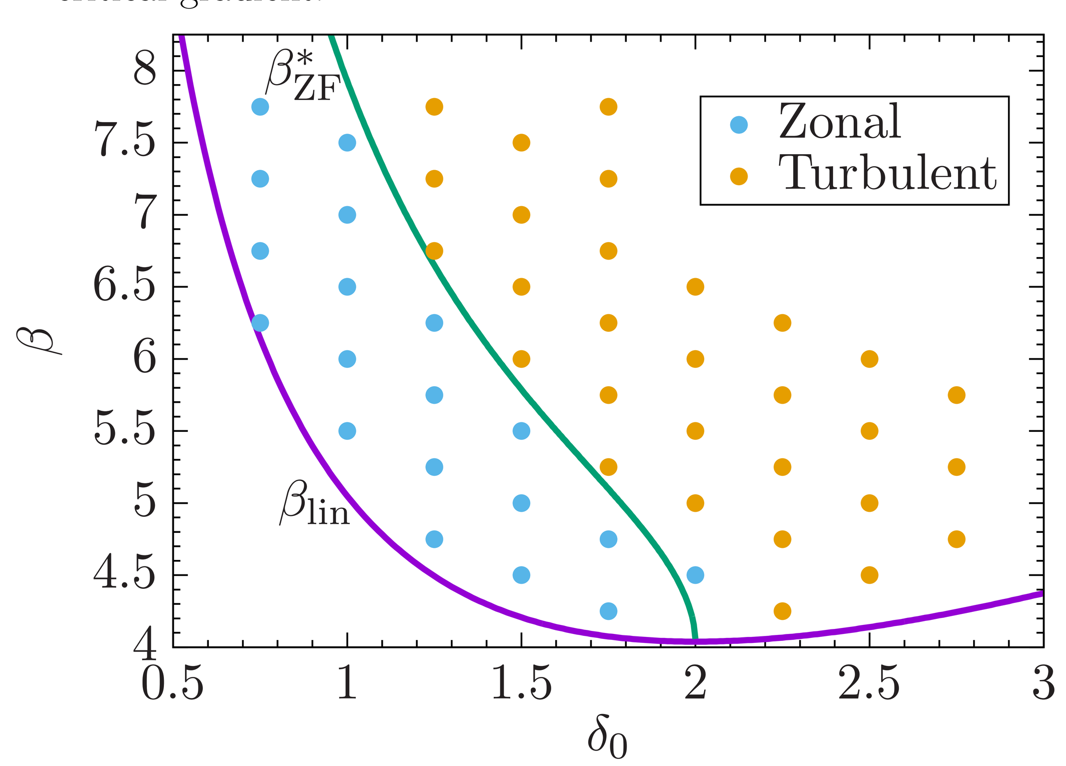


Figure 2: Parameter scan of the nonlinear system. The purple line marks the linear stability threshold, while the green line denotes the predicted end of the Dimits shift $\beta^*$ rom the analytic model.

**For a different approach based on wave kinetics, see Hongxuan Zhu's poster in tomorrow's session.**

## Would a Simpler Model Suffice?

The (modified) Terry-Horton equation includes the $\boldsymbol{E}\times\boldsymbol{B}$ nonlinearity $\{\varphi, \mathrm{i}\widehat{\delta}\varphi\}$ due to non-adiabatic electron response.

Neglecting this term leads to solutions that do not saturate beyond the Dimits shift!

Instead, monopolar and dipolar vortices (modons) form that, rather than getting sheared away, grow without bound.

**Equation for Zonal flow velocity $U$ with $\mathrm{i}\widehat{\delta} = \delta_0\partial_y$:**

$$\frac{\partial U}{\partial t} = \underbrace{-\frac{\partial}{\partial x}\langle v_x v_y\rangle}_{\text{Reynolds stress}} + \delta_0 \underbrace{\left(\langle v_x^2\rangle - \frac{1}{L_x}\int_0^{L_x}\mathrm{d}x\,\langle v_x^2\rangle\right)}_{\text{Terry-Horton stress}}$$

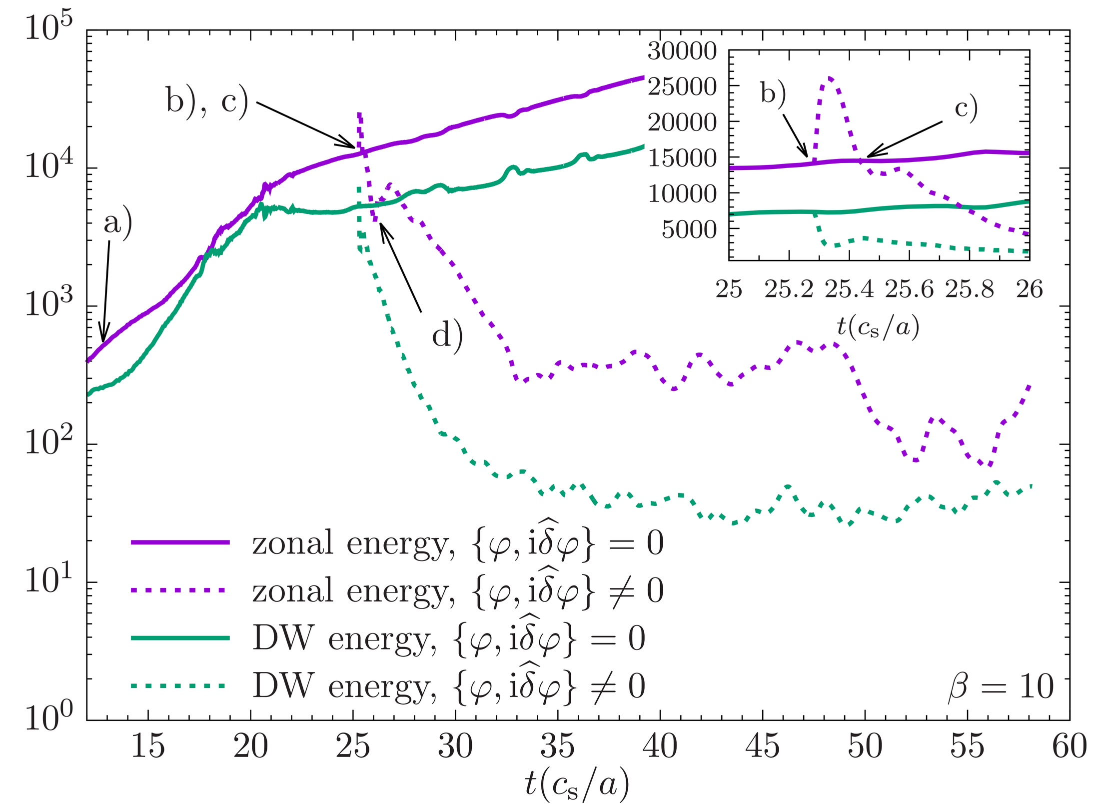


Figure 3: Evolution of the drift-wave and zonal energies for a simulation beyond the Dimits shift without $\{\varphi, \mathrm{i}\widehat{\delta}\varphi\}$ (solid line), along with a restarted simulation with $\{\varphi, \mathrm{i}\widehat{\delta}\varphi\}$ reinstated (dotted line).

## Spectral Transfer

The equation of motion in Fourier space is

$$\frac{\partial \varphi_{\boldsymbol{k}}}{\partial t} - \mathcal{L}\varphi_{\boldsymbol{k}} = \frac{1}{2\Gamma_{\boldsymbol{k}}}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2}\delta_{\boldsymbol{k},\boldsymbol{k}_1+\boldsymbol{k}_2}\hat{\boldsymbol{z}}\boldsymbol{\cdot}\boldsymbol{k}_1\times\boldsymbol{k}_2\,\varphi_{\boldsymbol{k}_1}\varphi_{\boldsymbol{k}_2}(\Gamma_{\boldsymbol{k}_2} - \Gamma_{\boldsymbol{k}_1}),$$

where $\Gamma_{\boldsymbol{k}} \doteq \widehat{\alpha}_{\boldsymbol{k}} - \mathrm{i}\widehat{\delta}_{\boldsymbol{k}} + k^2$ and $\mathcal{L}$ is the linear operator. Multiplying by $\varphi_{\boldsymbol{k}}^*$ leads to the spectral transfer function

$$\mathcal{T}(\boldsymbol{k}\,|\,\boldsymbol{p}) \doteq \frac{1}{4}\mathrm{Re}\left(\hat{\boldsymbol{z}}\boldsymbol{\cdot}\boldsymbol{p}\times\boldsymbol{k}\,\varphi_{\boldsymbol{p}}\varphi_{\boldsymbol{k}-\boldsymbol{p}}\varphi_{\boldsymbol{k}}^*\frac{\Gamma_{\boldsymbol{k}-\boldsymbol{p}} - \Gamma_{\boldsymbol{p}}}{\Gamma_{\boldsymbol{k}}}\right),$$

which denotes the spectral transfer from $\boldsymbol{k}$ to $\boldsymbol{p}$ through the mode with wavenumber $\boldsymbol{k}-\boldsymbol{p}$.

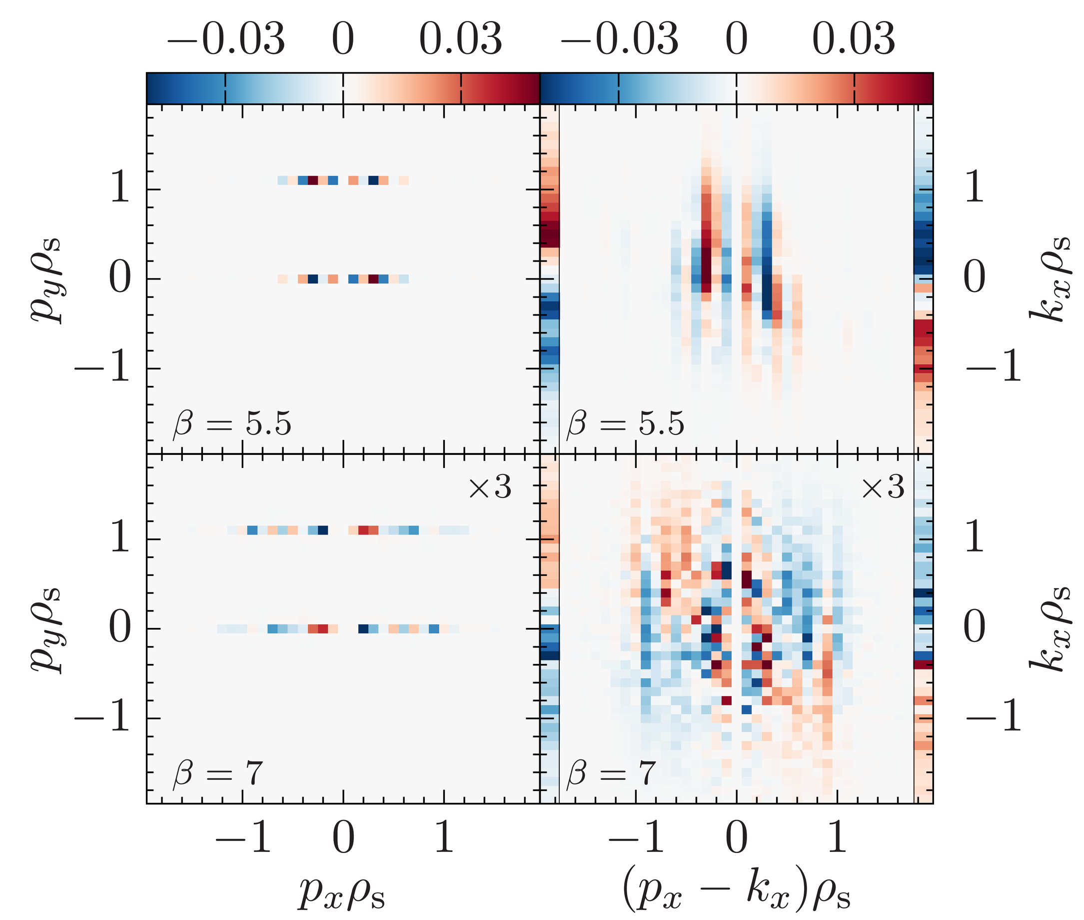


Figure 4: Time-averaged $\mathcal{T}(\boldsymbol{k}\,|\boldsymbol{p})$ for $k_y\rho_s = 1.1$ with (left) $k_x = 0$ and varied $p_x$, $p_y$; and (right) $p_y = k_y$ and varied $k_x$, $p_x$. Left (right) vertical colorbars in second column are $p_x$-averaged transfer for $p_x < k_x$ ($p_x > k_x$). Top (bottom) row denotes system in (beyond) the Dimits shift regime.

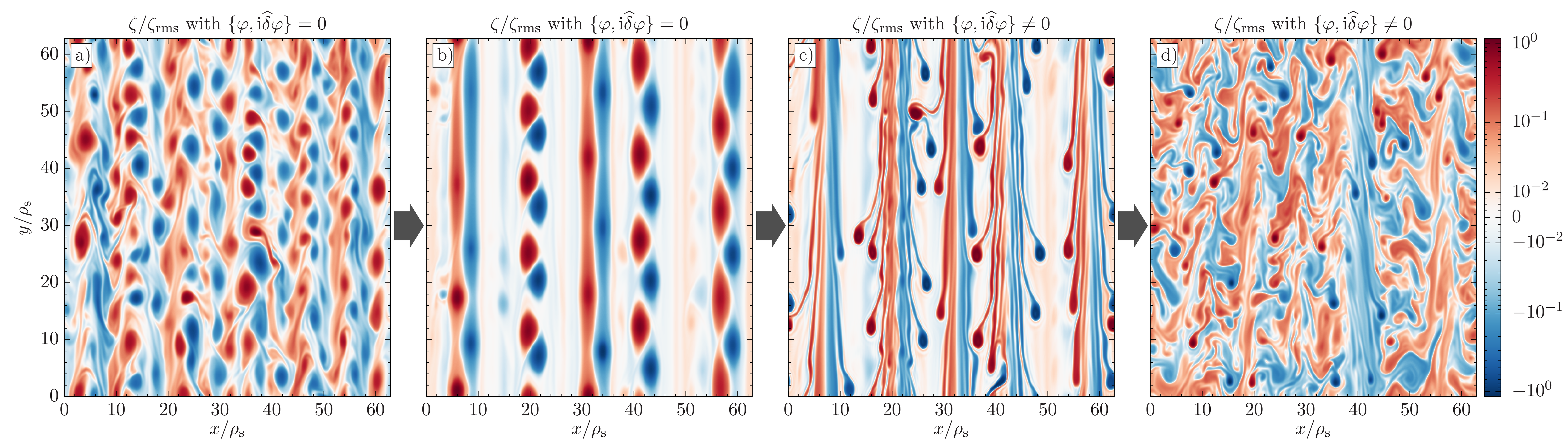


Figure 5: Pseudocolor of the modified vorticity scales by its RMS value ($\zeta/\zeta_{\mathrm{rms}}$) on a symmetric logarithmic plot. (Minor ticks around 0 denote linear region of scale.) Frame labels a–d denote times in figure 3. Vortices appear to dominate in the $\{\varphi, \mathrm{i}\widehat{\delta}\varphi\} = 0$ simulation, whereas the Terry-Horton stress destroys them, leading to an overall decrease in energy.

## CONCLUSIONS

- The modified Terry-Horton equation is the simplest system that exhibits a complete Dimits shift encountered thus far.
- The modified Hasegawa-Mima equation supports monopolar and dipolar vortices, while the modified Terry-Horton system shears them away.
- The spectral transfer function highlights the importance of coherency for zonal interactions during the Dimits shift.

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