Analytical Solution of the Grad Shafranov equation in an elliptical prolate Geometry

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Ancient attempt of fusion control

... even Math has an hearth
Let’s to restart from the Equilibrium equation

\[-\Delta^* \psi = \mu_0 R^2 P'(\psi) + II'(\psi) = \mu_0 R J \varphi\]

The operator $\Delta^*$ has the same structure (but a sign) of a normal Laplacian operator

\[
\Delta^* \psi = \frac{\partial^2 \psi}{\partial R^2} - \frac{1}{R} \frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial z^2} = \mu_0 R J \varphi
\]

\[
\Delta f = \frac{\partial^2 f}{\partial R^2} + \frac{1}{R} \frac{\partial f}{\partial R} + \frac{\partial^2 f}{\partial z^2}
\]
The Laplace equation $\Delta \Phi = 0$, for a given system of coordinates is said “separable” or “$R$ separable” when it admits solutions of the type

$$\Phi(u_1, u_2, ..., u_n) = U^1(u_1)U^2(u_2)....U^n(u_n)$$

In the case of rotational coordinates (like the toroidal one) the possibly admitted solution is the “$R$ separable”

$$\Phi(u_1, u_2, ..., u_n) = \frac{U^1(u_1)U^2(u_2)....U^n(u_n)}{R(u_1, u_2, ..., u_n)}$$

A necessary condition to get this is that

$$\sqrt{g_i} = R^2 f_i(u_1)F(u_1, u_2, ..., u_n)$$

Where $g_i$ are the metric elements
Let me to introduce the Cap-Cyclides Coordinates System

$$R + iZ = \frac{1}{a_s \cdot sn(\mu + iv) + i a_s k^{-0.5}} + \frac{i}{2a_s k^{-0.5}}$$

With

$$0 \leq \mu \leq K \quad 0 \leq \nu \leq K' \quad 0 \leq \phi \leq 2\pi$$

And $K$ and $K'$ are the complete elliptic integral with respectively modules $k$ and $k'$
Let me introduce the Cap-Cyclides Coordinates System

\[
\begin{align*}
    x & = \frac{\Lambda}{a_s \Gamma} dn(\nu, k') sn(\mu, k) \cos \varphi \\
    y & = \frac{\Lambda}{a_s \Gamma} dn(\nu, k') sn(\mu, k) \sin s \varphi \\
    z & = \frac{k^{0.5}}{2a_s \Gamma} \Pi
\end{align*}
\]

Where

\[
\begin{align*}
    \Lambda & = 1 - dn^2(\mu) sn^2(\nu) \\
    \Gamma & = sn^2(\mu) dn^2(\nu) + [(\Lambda/k^{0.5}) + cn(\mu) dn(\mu) sn(\nu) cn(\nu)]^2 \\
    \Pi & = (\Lambda^2/k) - [sn^2(\mu) dn^2(\nu) + cn(\mu)^2 dn(\mu)^2 sn(\nu)^2 cn(\nu)^2]
\end{align*}
\]
For the elliptical integral module $k \to 1$ and/or for the “radial” coordinate $\mu \to 0$ ($R \to \infty$) the coordinates goes to the circular toroidal one.
Center of Cap-Cyclide coordinate system is:

\[ R_0 = k^{0.5}/2a_s \]

\[ Z_0 = \pm k^{0.5}(1-k)/[2a_s(1+k)] \]

\[ X_0 = k^{0.5}/[a_s(1+k)] \]

that tends to the center of the circular toroidal system for \( k \to 1 \)

\[ R_0 = 1/2a_s ; Z_0 = 0 \]

We can define the local aspect ratio \( a/R \) and the local elongation \( b/a \)
After some algebra, we can get local aspect ratio

\[ R = \frac{k^{0.5} \left( k \, sn^2 \mu + 1 \right)}{2a_s (1 + k) \, sn \, \mu} \]

\[ a = \frac{k^{0.5} \, cn \, \mu \, dn \, \mu}{2a_s (1 + k) \, sn \, \mu} \]

For the large aspect ratio case \( \mu \to K \) we have

\[ R = R_0 = \frac{k^{0.5}}{2a_s} ; \quad a = 0 \]

Instead for \( \mu \to 0 \) for \( \kappa \to 1 \) we get

\[ \frac{R}{a} = \frac{(k \, sn^2 \mu + 1)}{cn \, \mu \, dn \, \mu} \approx \cosh (\mu) \]

Note that this behaviour is similar to the toroidal limit where

\[ R_{tor} / a_{tor} = \cosh \mu \]
The evaluation of the local elongation is a bit more complicated. We proceed by the fact that the maximum and minimum zeta value happens when, respectively, the two internal/external up and down radial coordinates coincides.

Eventually, after some heavy algebra, we get

$$z_{\text{max}} = b = \frac{k^{0.5}(1 - k \sin^2 \mu)}{2a_s(1 + k) \sin \mu} \implies b \frac{1}{a} = \frac{(1 - k \sin^2 \mu)}{\cos \mu \sin \mu}$$

$$\implies \mu = 0 \ (k = 0; k = 1) \implies \frac{b}{a} = 1 \quad \text{or} \quad \mu = K \ (k = 0; k \neq 1) \implies \frac{b}{a} = \infty$$

$$\implies \mu = K \ (k = 1) \implies \frac{b}{a} = 1$$
Laplace equation for an Euclidean space and for an orthogonal system is

$$\frac{1}{\sqrt{g}} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \sqrt{g} \frac{\partial \phi}{g_{ii} \partial x_i} \right) = 0$$

That assuming

$$\phi = U_1(x_1)....U_n(x_n)$$

becomes

$$\sum_{i=1}^{n} \frac{1}{U_i} \frac{\partial}{\partial x_i} \left( \sqrt{g} \frac{\partial U_i}{g_{ii} \partial x_i} \right) = 0$$

That using the separability conditions can be rewritten as

$$\sum_{i=1}^{n} \frac{1}{g_{ii}} \left[ \frac{1}{f_i U_i} \frac{d}{dx_i} \left( f_i \frac{dU_i}{dx_i} \right) + \sum_{j=1}^{n} \alpha_{ij} \Phi_{ij}(x_i) \right] = 0$$

That is satisfied only when any single addend is equal 0

$$\frac{1}{f_i U_i} \frac{d}{dx_i} \left( f_i \frac{dU_i}{dx_i} \right) + \sum_{j=1}^{n} \alpha_{ij} \Phi_{ij}(x_i) = 0$$

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\[1\] P. Moon and D. E. Spencer, Jour. Franklin Inst., 253, (1952) 585
Coming back to the Laplace Equation we can write the solution as

$$\phi(\mu, \nu, \varphi) = \left( \frac{\Gamma}{\Lambda} \right)^{0.5} M(\mu)N(\nu)F(\varphi)$$

That gives the following set of independent equations:

$$\frac{d^2 M}{d\mu^2} + \frac{\text{cn} \mu}{\text{sn} \mu} \frac{d}{d\mu} \left( \frac{d M}{d\mu} \right) + \left[ k^2 \text{sn}^2 \mu - \alpha_2 - \alpha_3 \left( k^2 \text{sn}^2 \mu + \frac{1}{\text{sn}^2 \mu} \right) \right] M = 0$$

$$\frac{d^2 N}{d\nu^2} - \frac{k'^2 \text{sn} \nu \text{cn} \nu}{\text{dn} \nu} \frac{d}{d\nu} \left( \frac{d N}{d\nu} \right) + \left[ -d\nu^2 + \alpha_2 + \alpha_3 \left( d\nu^2 + \frac{k^2}{dn^2 \nu} \right) \right] N = 0$$

$$\frac{d^2 F}{d\varphi^2} + \alpha_3 F = 0$$
If in the equation for \( M(\mu) \) we substitute \( \text{sn}^2 \mu = z_1 \) we get

\[
\frac{d^2 M}{dz_1^2} + \frac{dM}{dz_1} \left\{ \left( 1 - k^2 z_1 \right) \left[ 2 \left( 1 - z_1 \right) - z_1 \right] - k^2 z_1 \left( 1 - z_1 \right) \right\} \frac{1}{2 z_1 \left( 1 - k^2 z_1 \right) \left( 1 - z_1 \right)} + \left[ \frac{k^2 z_1 - \alpha_2 - \alpha_3 \left( k^2 z_1 + \frac{1}{z_1} \right) }{z_1 \left( 1 - k^2 z_1 \right) \left( 1 - z_1 \right)} \right] \frac{M}{4} = 0
\]

That can be written as

\[
\frac{d^2 M}{dz_1^2} + \frac{1}{2} \left[ \frac{1}{z_1 - a_1} + \frac{1}{z_1 - a_2} + \frac{2}{z_1 - a_3} \right] \frac{dM}{dz_1} + \frac{1}{4} \left[ \frac{A_0 + A_1 z_1 + A_2 z_1^2}{(z_1 - a_1)(z_1 - a_2)(z_1 - a_3)^2} \right] M = 0
\]

With \( a_1 = 1 ; a_2 = 1/k^2 ; a_3 = 0 ; A_0 = -\alpha_3/k^2 ; A_1 = -\alpha_2/k^2 ; A_2 = 1 - \alpha_3 \)
If in the equation for the poloidal angular part \( N(\nu) \) we substitute \( dn^2 \nu = z_2 \) we get

\[
\frac{d^2 N}{dz_2^2} + \frac{dN}{dz_2} \left\{ \left( z_2 - k^2 \right)[2(1-z_2) - z_2] + z_2(1-z_2) \right\} + \frac{\left[ -z_2 + \alpha_2 + \alpha_3 \left( \frac{z_2 + \frac{k^2}{z_2}}{z_2} \right) \right] N}{4z_2(z_2 - k^2)(1-z_2)} = 0
\]

That can be written as

\[
\frac{d^2 N}{dz_2^2} + \frac{1}{2} \left[ \frac{-1}{1-z_2} + \frac{1}{z_2 - k^2} + \frac{2}{z_2} \right] \frac{dN}{dz_2} + \frac{1}{4} \left[ \frac{-z_2^2 + \alpha_2 z_2 + \alpha_3 z_2^2 + \alpha_3 k^2}{z_2^2(1-z_2)(z_2 - k^2)} \right] N = 0
\]
That is a particular case of the general Bôcher\textsuperscript{1} equation

\[ \frac{d^2 \tilde{Z}}{dz^2} + \frac{d\tilde{Z}}{dz} P(z) + Q(z) \tilde{Z} = 0 \]

Where

\[ P(z) = \frac{1}{2} \left[ \frac{m_1}{z - a_1} + \frac{m_2}{z - a_2} + \ldots + \frac{m_{n-1}}{z - a_{n-1}} \right] \]

\[ Q(z) = \frac{1}{4} \left[ \frac{A_0 + A_1 z + \ldots + A_{l} z^l}{(z_1 - a_1)^{m_1} (z_1 - a_2)^{m_2} \ldots (z_1 - a_3)^{m_{n-1}}} \right] \]

With  \( a_1 = 1 \);  \( a_2 = 1/k^2 \);  \( a_3 = 0 \);  \( A_0 = -\alpha_3/k^2 \);  \( A_1 = -\alpha_2/k^2 \);  \( A_2 = 1 - \alpha_3 \)

\textsuperscript{1}M. Bôcher, "Über die Reihenentwickelungen der Potential theorie", Göttingen, B. G. Teubner., 1891
These equations are known as Wangerin\textsuperscript{1} equations

\[
\frac{d^2 \tilde{Z}}{dz^2} + \frac{1}{2} \left[ \frac{1}{z-1} + \frac{1}{z-c} + \frac{2}{z} \right] \frac{d\tilde{Z}}{dz} + \frac{1}{4} \left[ -q^2 c - p^2 c + (1-q^2)z^2 \right] = 0
\]

This equation has three singularities with poles in \( z=1, c, 0 \) of the order \( \{1,1,2\} \). Another pole is for \( z \to \infty \) of order 2. Consequently the Wangerin equation is characterized by poles \( \{1,1,2,2\} \). The equation admits a solution that depends from \( z, c, p, q \) and the general solution can be written as

\[
\phi(\mu,\nu,\varphi) = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \left[ A_p^0 D_p^q (k, sn^2 \mu) + B_p^0 \mathcal{F}_p^q (k, sn^2 \mu) \right] \left[ A_p^1 D_p^q (1/k, dn^2 \nu) + B_p^1 \mathcal{F}_p^q (1/k, dn^2 \nu) \right]
\]

\textsuperscript{1}A. Wangerin, "Theorie des Potentials und der Kugelfunktionen", Leipzig, G. J. Göschen'sche, 1909
We can use a complete different approach that will bring to an equation of the Wangerin type.

\[
(\Delta A)_\varphi = -\left(\nabla^*\nabla\right)\Psi / R
\]

Where \( \nabla^* \) is the Grad-Shafranov operator.

In axis symmetry the symmetrical component of the Laplacian operator is

\[
\frac{\partial^2 A_\varphi}{\partial u_1^2} + \frac{\partial^2 A_\varphi}{\partial u_2^2} + \frac{1}{2g_{33}} \left( \frac{\partial g_{33}}{\partial u_1} \frac{\partial A_\varphi}{\partial u_1} + \frac{\partial g_{33}}{\partial u_2} \frac{\partial A_\varphi}{\partial u_2} \right) + \\
+ \frac{A_\varphi}{2g_{33}} \left\{ \left( \frac{\partial^2 g_{33}}{\partial u_1^2} + \frac{\partial^2 g_{33}}{\partial u_2^2} \right) - \frac{1}{g_{33}} \left[ \left( \frac{\partial g_{33}}{\partial u_1} \right)^2 + \left( \frac{\partial g_{33}}{\partial u_2} \right)^2 \right] \right\} = 0
\]
In our Cap-Cyclide coordinates this become

\[
\frac{\partial}{\partial \mu} \left[ \frac{\partial A_\varphi}{\partial \mu} + \left( \frac{1}{f_1} \frac{\partial f_1}{\partial \mu} - R^2 \frac{\partial R^{-2}}{\partial \mu} \right) A_\varphi \right] + \frac{\partial}{\partial \nu} \left[ \frac{\partial A_\varphi}{\partial \nu} + \left( \frac{1}{f_2} \frac{\partial f_2}{\partial \nu} - R^2 \frac{\partial R^{-2}}{\partial \nu} \right) A_\varphi \right] = 0
\]

Where here \( R=(\Lambda/a_s \Gamma)^{1/2} \) is the quasi separation factor of the Laplace equation and \( f_1(\mu) = \text{sn}\mu, \quad f_2(\nu) = \text{dn}\nu \)

This equation is equal to the Laplace equation PLUS a non differential term in \( A_\varphi \)
In our case of the Caps-Cyclide coordinates this becomes

\[
\left[ \frac{\partial}{\partial \mu} \left( \frac{1}{\sin \mu} \frac{\partial s \mu}{\partial \mu} - \frac{\Lambda}{\Gamma} \frac{\partial}{\partial \mu} \frac{\Gamma}{\Lambda} \right) + \frac{\partial}{\partial \nu} \left( \frac{1}{\sin \nu} \frac{\partial s \nu}{\partial \nu} - \frac{\Lambda}{\Gamma} \frac{\partial}{\partial \nu} \frac{\Gamma}{\Lambda} \right) \right] = \\
= -\left( k^2 \sin^2 \mu + \frac{1}{\sin^2 \mu} \right) + \left( \sin^2 \nu + \frac{k^2}{\sin^2 \nu} \right)
\]

We know that the solution for the Vector Laplace equation in axis symmetry is separable and can be written as

\[
A_\varphi(\mu, \nu) = \frac{M(\mu)N(\nu)}{R(\mu, \nu)} = \left( \frac{\Gamma}{\Lambda} \right)^{1/2} M(\mu)N(\nu)
\]
Consequently it can be written a system of independent equation similar to the Laplace ones with $q^2=\alpha_3=0$ (axis symmetry), but with an additional terms in $A\varphi$

\[
\frac{d^2 M}{d\mu^2} + \frac{c n \mu}{s n \mu} \frac{d n \mu}{d\mu} \frac{dM}{d\mu} + \left( k^2 s n^2 \mu - \alpha_2 - k^2 s n^2 \mu + \frac{1}{s n^2 \mu} \right) M = 0
\]

\[
\frac{d^2 N}{d\nu^2} - k^2 s n \nu \frac{c n \nu}{d\nu} \frac{dN}{d\nu} + \left( -d n^2 \nu + \alpha_2 + d n^2 \nu + \frac{k^2}{s n^2 \nu} \right) N = 0
\]

But this is exactly the set of equations for the 3D scalar Laplace equation with $q^2=\alpha_3=1$

So the solution of the axial component of the vectorial Laplace equation for the vector potential $A$ is the same solution that for the scalar Laplace equation with $q=1$ and $\varphi=0$
If now we remember that the flux function is defined as

$$\Psi = \oint A_\varphi \, d\ell$$

We have that

$$\Psi = 2\pi \ g_{33}^{1/2} \ A_\varphi(\mu, \nu) = 2\pi \ g_{33}^{1/2} \ \frac{M(\mu)N(\nu)}{R(\mu, \nu)} = 2\pi \ g_{33}^{1/2} \left( \frac{\Gamma}{\Lambda} \right)^{1/2} \ M(\mu)N(\nu)$$

And eventually

$$\Psi(\mu, \nu) = \frac{2\pi}{a_s} \left( \frac{\Lambda}{\Gamma} \right)^{1/2} \ sn \mu \ dn \nu \left[ A_p^0 D_\nu^1(k, sn^2 \mu) + B_p^0 \mathcal{F}_p^1(k, sn^2 \mu) \right] \left[ A_p^1 D_\nu^1(1/k, dn^2 \nu) + B_p^1 \mathcal{F}_p^1(1/k, dn^2 \nu) \right]$$
Conclusion

Mixing up some different music styles, i.e. playing with the math the analytical solution of Grad-Shafranov Equation has been written in a toroidal elliptical prolate geometry. This will allow to develop a reconstructive equilibrium code based on the natural elongated plasma geometry.