

Wave Equation for Cold Plasma for Heating and Diagnostics Applications in toroidal confined plasmas

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Abstract

- The solution of the electromagnetic wave equation in toroidal plasmas in time-space domain is particularly difficult also when the cold plasma approximation is invoked, this is due essentially to the fact that the equation cannot be recast to the Helmholtz equation owing to the presence of the “curl curl” differential operator. Another difficulty is related to the fact that the toroidal geometry introduces unpleasant metric coefficients. Solutions based on asymptotic techniques (WKB) have proved useful and effective [1]. In this work we focus on the possibility of a complete solution of the equation in an infinite half-plane for a harmonic perturbation using the Fourier and Laplace transform for the fields. The Laplace transform on space applies on the radial dimension, and allows us to set the appropriate boundary conditions of the field at the edge. In order to solve the wave vector equation, a new dispersion relation is derived and studied in detail. In some relevant cases (e.g. Lower Hybrid Waves) the longitudinal component of the field can even become the largest one, this means that the wave electric field can be considered irrotational and the wave equation becomes the more manageable Poisson’s equation. This analysis is particularly useful in order to predict, for example, the behavior of a broadband terahertz (THz) pulse in a quasi-uniform and dispersive plasma, or studying the Current Drive of the LHW with a possible application to reactor plasma like ITER and DEMO.
- [1] S. Yu. Dobrokhotov, A. Cardinali, A. I. Klevin, and B. Tirozzi, Doklady Mathematics, 94 (2016), ISSN 1064-5624.

Wave equation for cold plasmas (fluid approach)

$$c^2 \nabla \wedge \nabla \wedge \vec{E}(\vec{r}, t) + \frac{\partial^2 \vec{E}(\vec{r}, t)}{\partial t^2} + 4\pi \sum_{\alpha=e,i} \rho_\alpha \frac{\partial \vec{v}_\alpha}{\partial t} = 0$$

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot \rho_\alpha \vec{v}_\alpha = 0$$

$$m_\alpha \rho_\alpha \frac{\partial \vec{v}_\alpha}{\partial t} = q_\alpha \rho_\alpha \left(\vec{E} + \frac{\vec{v}_\alpha \wedge \vec{B}}{c} \right) - q_\alpha \nabla p_\alpha$$

Integro-Differential equation system

Linearization of the equations

$$c^2 \nabla \wedge \nabla \wedge \vec{E}(\vec{r}, t) + \frac{\partial^2 \vec{E}(\vec{r}, t)}{\partial t^2} + 4\pi \sum_{\alpha=e,i} \left(\rho_\alpha^{(0)} \frac{\partial \vec{v}_\alpha^{(1)}}{\partial t} + \cancel{\rho_\alpha^{(1)} \frac{\partial \vec{v}_\alpha^{(0)}}{\partial t}} \right) = 0$$

$$\frac{\partial \rho_\alpha^{(0)}}{\partial t} + \nabla \cdot \rho_\alpha^{(0)} \vec{v}_\alpha^{(0)} = 0 \Rightarrow \frac{D \rho_\alpha^{(0)}}{Dt} + \cancel{\rho_\alpha^{(0)} \nabla \cdot \vec{v}_\alpha^{(0)}} = 0 \Rightarrow \rho^{(0)} = \text{const}$$

$$\frac{\partial \rho_\alpha^{(1)}}{\partial t} + \nabla \cdot \left[\rho_\alpha^{(0)} \vec{v}_\alpha^{(1)} + \cancel{\rho_\alpha^{(1)} \vec{v}_\alpha^{(0)}} \right] = 0 \Rightarrow \frac{\partial \rho_\alpha^{(1)}}{\partial t} + \rho_\alpha^{(0)} \nabla \cdot \vec{v}_\alpha^{(1)} = 0$$

$$m_\alpha \rho_\alpha^{(0)} \frac{\partial \vec{v}_\alpha^{(1)}}{\partial t} = q_\alpha \rho_\alpha^{(0)} \left(\vec{E} + \frac{\vec{v}_\alpha^{(1)} \wedge \vec{B}_0}{c} \right) - \cancel{q_\alpha \nabla p_\alpha^{(1)}} \Rightarrow \text{cold plasma}$$

Merging and Integro-Differential Wave Equation

$$c^2 \nabla \wedge \nabla \wedge \vec{E}(\vec{r}, t) + \frac{\partial^2 \vec{E}(\vec{r}, t)}{\partial t^2} + 4\pi \sum_{\alpha=e,i} \left(\rho_\alpha^{(0)} \frac{\partial \vec{v}_\alpha^{(1)}}{\partial t} \right) = 0$$

$$\dot{\vec{v}}_\alpha^{(1)}(t) = \Omega_{c\alpha} \hat{\underline{\Omega}}_\alpha \cdot \vec{v}_\alpha^{(1)}(t) + \vec{g}(t)$$

$$\hat{\underline{\Omega}}_\alpha \cdot \vec{v}_\alpha^{(1)}(t) = \sum_{i,j,k=1}^3 \epsilon_{ijk} \vec{e}_i v_j b_k$$

$$\Omega_{c\alpha} = \frac{q_\alpha |B|}{cm_\alpha}$$

Diagonalization of $\hat{\underline{\Omega}}_\alpha$ and solution of Eqs for the velocity

$$q_k = e^{\lambda_k t} \int_0^t e^{-\lambda_k s} g_k(s) ds + c_k e^{\lambda_k t} \Rightarrow k = 1, 2, 3$$

$\vec{v}_k^{(1)} = T_{k,m} q_m$ $\frac{T}{\lambda_k}$ is the nonsingular transform matrix whose columns are the eigenvectors of $\hat{\underline{\Omega}}_\alpha$
 λ_k are the eigenvalues of $\hat{\underline{\Omega}}_\alpha$

Boundary-initial conditions

$$\vec{E}(\vec{r}, t=0) = \Psi(\vec{r})$$

$$\vec{E}(\vec{r}_0, t) \Big|_{\partial\Sigma} = \vec{\Upsilon}(t)$$

$\Psi(\vec{r})$ **Can be taken as Gaussian beam**

Incoming wave

Harmonic field steady state

$$\vec{E}(\vec{r}_0, t) \Big|_{\partial\Sigma} = \Upsilon(t) = e^{-i\omega t}$$

Wave equation in space: Vector *Helmholtz* equation

$$c^2 \nabla \wedge \nabla \wedge \vec{E}(\vec{r}) = \omega^2 \left(\underline{\underline{I}} - \frac{i}{\omega} \sum_{\alpha=e,i} \omega_{p\alpha}^2 \underline{\underline{C}}_\alpha^{-1} \right) \cdot \vec{E}(\vec{r})$$

$$\underline{\underline{C}}_\alpha^{-1} = \frac{1}{-i\omega(\omega^2 - \Omega_{c\alpha}^2)} \begin{bmatrix} (i\omega)^2 + \Omega_{c\alpha}^2 (\hat{b}_1)^2 & -(i\omega \Omega_{c\alpha} \hat{b}_3) + \Omega_{c\alpha}^2 (\hat{b}_1 \hat{b}_2) & (i\omega \Omega_{c\alpha} \hat{b}_2) + \Omega_{c\alpha}^2 (\hat{b}_1 \hat{b}_3) \\ (i\omega \Omega_{c\alpha} \hat{b}_3) + \Omega_{c\alpha}^2 (\hat{b}_1 \hat{b}_2) & (i\omega)^2 + \Omega_{c\alpha}^2 (\hat{b}_2)^2 & -(i\omega \Omega_{c\alpha} \hat{b}_1) + \Omega_{c\alpha}^2 (\hat{b}_2 \hat{b}_3) \\ -(i\omega \Omega_{c\alpha} \hat{b}_2) + \Omega_{c\alpha}^2 (\hat{b}_1 \hat{b}_3) & (i\omega \Omega_{c\alpha} \hat{b}_1) + \Omega_{c\alpha}^2 (\hat{b}_2 \hat{b}_3) & (i\omega)^2 + \Omega_{c\alpha}^2 (\hat{b}_3)^2 \end{bmatrix}$$

$$\underline{\underline{\varepsilon}} = \left(\underline{\underline{I}} - \frac{i}{\omega} \sum_{\alpha=e,i} \omega_{p\alpha}^2 \underline{\underline{C}}_\alpha^{-1} \right)$$

Cartesian Geometry

$$\begin{aligned}
 -\frac{\partial}{\partial z} \left\{ \left[\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right] \right\} + \frac{\partial}{\partial y} \left\{ \left[\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right] \right\} - \frac{\omega^2}{c^2} \left[\varepsilon_{xx} E_x - \varepsilon_{xy} (\hat{b}_y E_z - \hat{b}_z E_y) \right] &= 0 \\
 -\frac{\partial}{\partial x} \left\{ \left[\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right] \right\} + \frac{\partial}{\partial z} \left\{ \left[\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right] \right\} - \frac{\omega^2}{c^2} \left[-\varepsilon_{xy} \hat{b}_z E_x + \varepsilon_{xx} E_y - (\varepsilon_{xx} - \varepsilon_{zz}) \hat{b}_y^2 E_y \right] &= 0 \\
 -\frac{\partial}{\partial y} \left\{ \left[\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right] \right\} + \frac{\partial}{\partial x} \left\{ \left[\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right] \right\} - \frac{\omega^2}{c^2} \left[\varepsilon_{xy} \hat{b}_y E_x + \varepsilon_{xx} E_z - (\varepsilon_{xx} - \varepsilon_{zz}) \hat{b}_z^2 E_z \right] &= 0
 \end{aligned}$$

We consider now that all the coefficients are constant and the Dirichlet boundary conditions on the surface S (infinite half-plane $x=0$)

$$\vec{E}(x=0, y, z) = \vec{g}(y, z) = \vec{e} E_0 e^{-\frac{z^2}{2\Delta z^2}} e^{-\frac{y^2}{2\Delta y^2}}$$

Cartesian Geometry

we can apply the Fourier Transform of the field in the y-z direction

$$\vec{E}(x,y,z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk_y \int_{-\infty}^{+\infty} dk_z \vec{E}(x, k_z) e^{ik_y y} e^{ik_z z} \delta(k_y)$$

Fourth order equation

$$A \frac{d^4 E_z}{dx^4} + B \frac{d^2 E_z}{dx^2} + C E_z = 0$$

$$A = 1$$

$$B = - \left[\frac{(k_z^2 - k_0^2 S)(P + S) + k_0^2 D^2}{S} \right]$$

$$C = \frac{P \left[(k_z^2 - k_0^2 S)^2 - k_0^4 D^2 \right]}{S}$$

$$k_0^2 = \left(\frac{\omega}{c} \right)^2$$

Laplace Transform Solution I

$$\mathcal{O}\left(E_z(x, k_z)\right) = \mathfrak{I}(s) \equiv \int_0^{\infty} e^{-sx} E_z(x, k_z) dx$$

$$\mathfrak{I}(s) = \frac{s(s^2 + B)y_0}{s^4 + Bs^2 + C} + \frac{sy_0''}{s^4 + Bs^2 + C} + \frac{y_0'''}{s^4 + Bs^2 + C}$$

and

$$E_z(x, k_z) = \mathcal{O}^{-1}(\mathfrak{I}(s)) = \frac{1}{2\pi i} \lim_{x \rightarrow \infty} \int_{\gamma - ix}^{\gamma + ix} e^{-sx} E_z(s, k_z) ds$$

Laplace Transform Solution II

$$\begin{aligned}E_z(x, k_z) = & \frac{y_0 \left[(q_S^2 + B)(e^{q_S x} + e^{-q_S x}) - (q_F^2 + B)(e^{q_F x} + e^{-q_F x}) \right]}{2(q_S - q_F)(q_S + q_F)} + \\& + \frac{y_0'' \left[(e^{q_S x} + e^{-q_S x}) - (e^{q_F x} + e^{-q_F x}) \right]}{2(q_S - q_F)(q_S + q_F)} + \\& + \frac{y_0'''}{2(q_S - q_F)(q_S + q_F)} \left[\frac{(e^{q_S x} - e^{-q_S x})}{q_S} - \frac{(e^{q_F x} + e^{-q_F x})}{q_F} \right]\end{aligned}$$

Gaussian wave-packet at the boundary x=0

$$y_0 = E_0(0) e^{-\frac{z^2}{2\Delta z^2}} e^{-\frac{y^2}{2\Delta y^2}}$$

$$y_0'' = -\frac{1}{\Delta x^2} E_0(0) e^{-\frac{z^2}{2\Delta z^2}} e^{-\frac{y^2}{2\Delta y^2}}$$

$$y_0''' = \frac{1}{\Delta x^4} E_0(0) e^{-\frac{z^2}{2\Delta z^2}} e^{-\frac{y^2}{2\Delta y^2}}$$

Fourier transform of the field obtained above

$$\begin{aligned}
E_z(x, z) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk_z \vec{E}(x, k_z) e^{ik_z z} = \\
&= \frac{E_0(0)}{2\pi} e^{-\frac{z^2}{2\Delta z^2}} e^{-\frac{y^2}{2\Delta y^2}} \int_{-\infty}^{+\infty} dk_z \frac{\left[(q_s^2 + B)(e^{q_s x} + e^{-q_s x}) - (q_F^2 + B)(e^{q_F x} + e^{-q_F x}) \right]}{2(q_s - q_F)(q_s + q_F)} + \\
&\quad - \frac{E_0(0)}{2\pi \Delta x^2} e^{-\frac{z^2}{2\Delta z^2}} e^{-\frac{y^2}{2\Delta y^2}} \int_{-\infty}^{+\infty} dk_z \frac{\left[(e^{q_s x} + e^{-q_s x}) - (e^{q_F x} + e^{-q_F x}) \right]}{2(q_s - q_F)(q_s + q_F)} + \\
&\quad + \frac{E_0(0)}{2\pi \Delta x^4} e^{-\frac{z^2}{2\Delta z^2}} e^{-\frac{y^2}{2\Delta y^2}} \int_{-\infty}^{+\infty} dk_z \frac{1}{2(q_s - q_F)(q_s + q_F)} \left[\frac{(e^{q_s x} - e^{-q_s x})}{q_s} - \frac{(e^{q_F x} + e^{-q_F x})}{q_F} \right]
\end{aligned}$$

Poles, cut-offs and confluences

$$q_S^2 = \frac{\left[\frac{(k_z^2 - k_0^2 S)(P + S) + k_0^2 D^2}{S} \right] + \left| \frac{(k_z^2 - k_0^2 S)(P + S) + k_0^2 D^2}{S} \right| \sqrt{1 - 4 \frac{PS[(k_z^2 - k_0^2 S)^2 - k_0^4 D^2]}{[(k_z^2 - S)(P + S) + k_0^2 D^2]^2}}}{2}$$

$$q_F^2 = \frac{\left[\frac{(k_z^2 - k_0^2 S)(P + S) + k_0^2 D^2}{S} \right] - \left| \frac{(k_z^2 - k_0^2 S)(P + S) + k_0^2 D^2}{S} \right| \sqrt{1 - 4 \frac{PS[(k_z^2 - k_0^2 S)^2 - k_0^4 D^2]}{[(k_z^2 - k_0^2 S)(P + S) + k_0^2 D^2]^2}}}{2}$$

If $\Delta = 1 - 4 \frac{PS[(k_z^2 - k_0^2 S)^2 - k_0^4 D^2]}{[(k_z^2 - S)(P + S) + k_0^2 D^2]^2} > 0 \Rightarrow q_S^2 \& q_F^2 \in \Re$

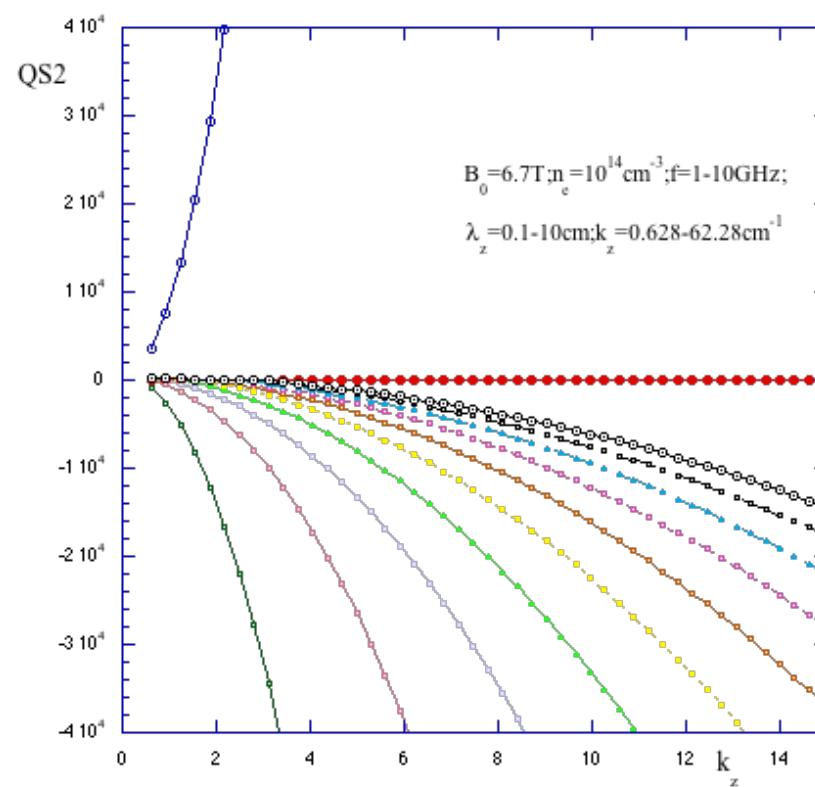
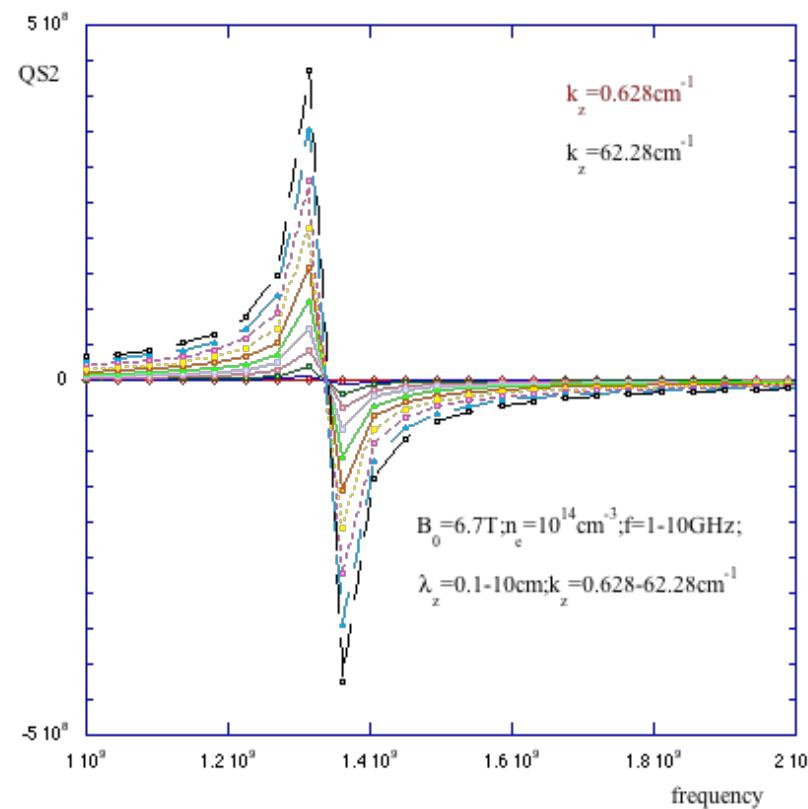
if

$q_S^2 \& q_F^2 > 0 \Rightarrow q_S \& q_F \in \Re \Rightarrow \text{no propagation}$

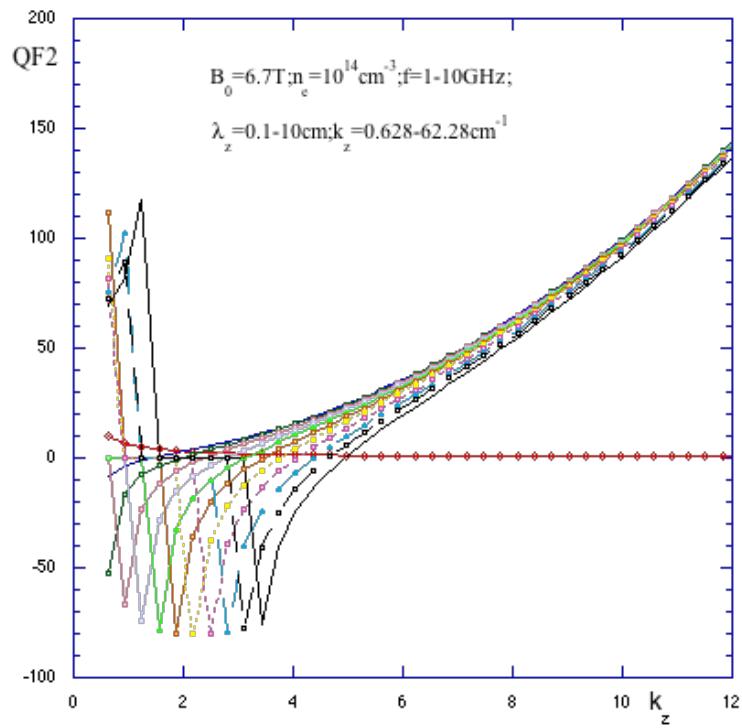
$q_S^2 \& q_F^2 < 0 \Rightarrow q_S \& q_F \in \Im \Rightarrow \text{propagation}$

$\Delta = 0 \Rightarrow q_S^2 = q_F^2 \Rightarrow \text{confluence}$

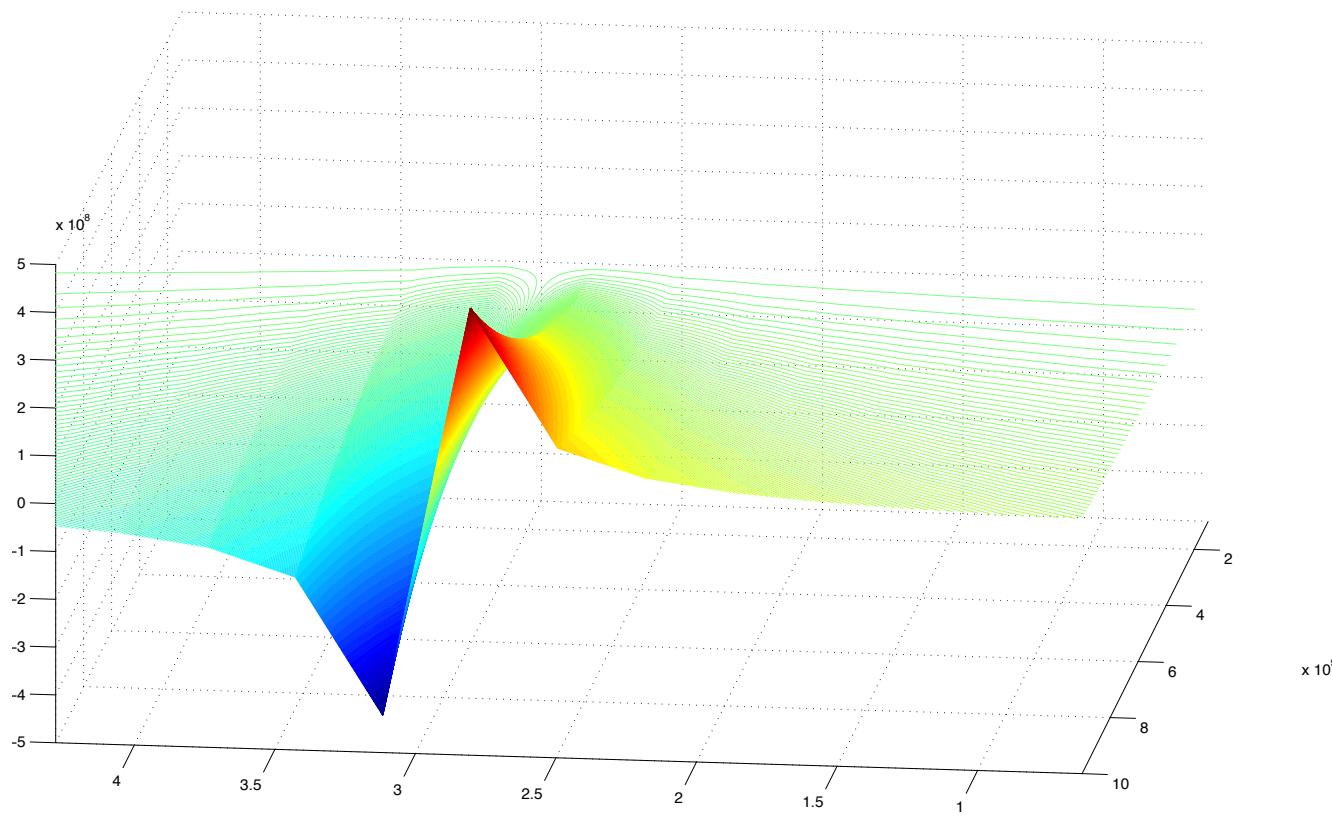
Poles of the argument of the Laplace transform as function of frequency and wavelengths



Poles of the argument of the Laplace transform as function of frequency and wavelengths



Poles surface in frequency and wavelength



Complete calculation $k_y \neq 0$

We make laplace transform in x and fourier in y and z

$$E(s, k_x, k_z) = \int_0^\infty dx \exp(-sx) \int dy dz \exp(ik_y y + ik_z z) E'(x, y, z) \quad (1)$$

where E' is a the three dimensional vector of the em field. making the transformation and taking the density constant we get that the equations are

$$\begin{pmatrix} y^2 + z^2 - \frac{\omega^2}{c^2} S & i \left(\frac{\omega^2}{c^2} D - sy \right) & -isz \\ -i \left(\frac{\omega^2}{c^2} D + sy \right) & -s^2 + z^2 - \frac{\omega^2}{c^2} S & -yz \\ -isz & -yz & -s^2 + y^2 - \frac{\omega^2}{c^2} P \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} i(k_z E_z^0 + k_y E_y^0) \exp(-\sigma_2^2 k_y^2/2 - \sigma_3^2 k_z^2/2) \\ (-s E_y^0 + i k_y E_x^0) \exp(-\sigma_2^2 k_y^2/2 - \sigma_3^2 k_z^2/2) \\ (-s E_z^0 + i k_z E_x^0) \exp(-\sigma_2^2 k_y^2/2 - \sigma_3^2 k_z^2/2) \end{pmatrix}$$

Complete calculation $k_y \neq 0$

The determinant of the matrix of the coefficient is

$$\begin{aligned} H = & -\frac{\omega^2}{c^2} S s^4 + \\ & + \left[-\left(z^2 + y^2 - \frac{\omega^2}{c^2} S \right)^2 + \frac{\omega^2}{c^2} \left(z^2 + y^2 - \frac{\omega^2}{c^2} S \right) P + z^4 + y^4 + 2z^2y^2 - \frac{\omega^2}{c^2} z^2S - \frac{\omega^2}{c^2} y^2P + \frac{\omega^4}{c^4} G^2 \right] s^2 + \\ & + \left[\left(-y^2 + \frac{\omega^2}{c^2} P \right) \left(-z^2 + \frac{\omega^2}{c^2} S \right) - z^2y^2 \right] \left(z^2 + y^2 - \frac{\omega^2}{c^2} S \right) + \frac{\omega^4}{c^4} \left(-y^2 + \frac{\omega^2}{c^2} P \right) G^2 \end{aligned}$$

The determinant = to zero determines the poles of the Laplace-transformed field

High frequency limit

- In the case

$$\left(\frac{1}{\vec{v}_\alpha^{(1)}} \cdot \frac{d\vec{v}_\alpha^{(1)}}{dt} \right) \sim \left(\frac{1}{\vec{E}} \cdot \frac{d\vec{E}}{dt} \right) \gg \Omega_{c\alpha}$$

- We have

$$\frac{\partial \vec{v}_\alpha^{(1)}}{\partial t} = \frac{q_\alpha \vec{E}(\vec{r}, t)}{m_\alpha}$$

$$c^2 \nabla \wedge \nabla \wedge \vec{E}(\vec{r}, t) + \frac{\partial^2 \vec{E}(\vec{r}, t)}{\partial t^2} + 4\pi \sum_{\alpha=e,i} \left(\rho_\alpha^{(0)} \frac{q_\alpha \vec{E}(\vec{r}, t)}{m_\alpha} \right) = 0$$

Helmholtz decomposition

$$\vec{E}(\vec{r}, t) = \vec{E}_t(\vec{r}, t) + \vec{E}_l(\vec{r}, t)$$

curl-free component of a vector field as the **longitudinal component**
divergence-free component as the **transverse component**

$$-c^2 \nabla^2 \vec{E}_t + \frac{\partial^2 \vec{E}_t}{\partial t^2} = - \sum_{\alpha=e,i} \omega_{p\alpha}^2(\vec{r}) \vec{E}_t$$

$$\left(\frac{\partial^2}{\partial t^2} + \sum_{\alpha=e,i} \omega_{p\alpha}^2(\vec{r}) \right) \vec{E}_l = 0$$

-Paraxial wave approximation of Gaussian beams

-Maslov theory: S. Yu. Dobrokhotov, A. Cardinali, A. I. Klevin, B. Tirozzi, Maslov Complex Germ and High-Frequency Gaussian Beams for Cold Plasma in a Toroidal Domain, Doklady Mathematics, 2016, Vol. 94, No. 2, pp. 1–6

Wave Equation in Cartesian Geometry

$$-\left(\nabla^2 \vec{E}\right)_x \equiv -\frac{\partial^2 E_x}{\partial x^2} - \frac{\partial^2 E_x}{\partial y^2} - \frac{\partial^2 E_x}{\partial z^2} = -\frac{1}{c^2} \frac{\partial^2 E_x}{\partial t^2} - \frac{\sum_{\alpha=e,i} \omega_{p\alpha}^2(\vec{r})}{c^2} E_x$$
$$-\left(\nabla^2 \vec{E}\right)_y \equiv -\frac{\partial^2 E_y}{\partial x^2} - \frac{\partial^2 E_y}{\partial y^2} - \frac{\partial^2 E_y}{\partial z^2} = -\frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2} - \frac{\sum_{\alpha=e,i} \omega_{p\alpha}^2(\vec{r})}{c^2} E_y$$
$$-\left(\nabla^2 \vec{E}\right)_z \equiv -\frac{\partial^2 E_z}{\partial x^2} - \frac{\partial^2 E_z}{\partial y^2} - \frac{\partial^2 E_z}{\partial z^2} = -\frac{1}{c^2} \frac{\partial^2 E_z}{\partial t^2} - \frac{\sum_{\alpha=e,i} \omega_{p\alpha}^2(\vec{r})}{c^2} E_z$$

Conclusions

- Attempt to deduce a wave equation in space-time from the fluid theory
 - Integro-Differential equation
 - Boundary-Initial value problem
- Harmonic field
 - Intermediate frequency \longrightarrow vector Helmholtz equation
 - Solution in an unbounded half plane using Laplace-Fourier Transform
 - Solution in the homogeneous case
 - Comparison with the asymptotic solution obtained in weak inhomogeneous case

Conclusions

- High Frequency limit
 - isotropy
 - Helmholtz decomposition
 - to deal with vector Laplacian operator