Relativistic Boltzmann collision operator for runaway-avalanche studies

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Hard-sphere collisions

Coulomb collisions
Boltzmann: \[
\frac{df_e}{dt}(p) = \int dp_1 \, n_e v_1 f_e(p_1) \frac{\partial \sigma}{\partial p}(p_1 \rightarrow p)
\]
\[
- n_e v_f(p) \int dp_1 \frac{\partial \sigma}{\partial p_1}(p \rightarrow p_1)
\]

Fokker-Planck: \[
\frac{df_e}{dt} = \frac{\partial}{\partial p} \cdot \left[ A f_e + \frac{\partial}{\partial p} \cdot (D f_e) \right]
\]
Electron runaway

Runaways are created in the presence of strong electric fields by primary (Dreicer) or secondary (avalanche, knock-on) generation.
Avalanche generation

Runaway growth rates:

**Knock-on**: \( \frac{\partial n_{RE}}{\partial t} \sim \mathcal{O} \left[ n_{RE} \left( \frac{E}{E_c} - 1 \right) \right] \)

**Primary**: \( \frac{\partial n_{RE}}{\partial t} \sim \mathcal{O} \left[ n_e \ln \Lambda \exp \left( -\frac{E_D}{4E} \right) \right] \)

where \( E_D = \left( \frac{m_e c^2}{T_e} \right) E_c \)

\[ \frac{\text{Knock-on}}{\text{Primary}} \sim \left[ \frac{1}{\ln \Lambda} \right] \]

BUT: Knock-on still wins for small \( E/E_D \) or large \( n_{RE}/n_e \)!

[Connor, Hastie NF 1975; Rosenbluth, Putvinski NF 1998]
Avalanche generation

To describe knock-on collisions we add a (simplified) Boltzmann operator:

$$\frac{df_e}{dt} = C_{FP}\{f_e\} + C_{boltz}\{f_e\}$$

Beware of double counting!

Generally we can linearize ($n_{RE} \ll n_e$)

$$C_{boltz}\{f_e, f_e\} \approx C_{boltz}\{f_e, f_e0\} + C_{boltz}\{f_e0, f_e\}.$$
Avalanche generation

The two most established knock-on models today:

\[ C_{\text{knock-on}} = C_{\text{boltz}} \{ n_e \delta(p), f_e \} \quad \text{(only field-particle term)} \]

Rosenbluth-Putvinski:

\[ f_e(p) = n_{\text{RE}} \lim_{p_0 \to \infty} \frac{1}{p^2} \delta(p - p_0) \delta(\cos \theta - 1) \]

Chiu-Harvey:

\[ f_e(p) = F(p) \delta(\cos \theta - 1) \]

\[ \left( F(p) = \int_{-1}^{1} f_e(p) \, d(\cos \theta) \right) \]

[Rosenbluth, Putvinski NF 1997; Chiu, Rosenbluth, Harvey NF 1998]
Avalanche generation

So how do these operators behave?
Avalanche generation

Both models have limitations:

- Double counting collisions
- Non-conservation of momentum and energy
  - Rosenbluth-Putvinski even creates infinite energy and momentum!
- Chiu-Harvey model ignores pitch-angle distribution
- Arbitrary cut-off affecting solutions
Avalanche generation

We solved this, by

- Accounting for full $f_e(p)$
- Including the test-particle term [restores conservation laws]
- Modify $\ln \Lambda$ in Fokker-Planck operator [avoids double counting]

[O. Embréus et al., APS 2015, PP12.00107; T. Fülöp et al., IAEA 2016, TH/P4-1]
Avalanche generation
The various knock-on operators have been implemented in the 0D+2P kinetic-equation solver CODE. The tool contains all momentum-space physics needed to describe runaway generation and decay:

- Primary, secondary and hot-tail generation from first principles
- Synchrotron and bremsstrahlung radiation losses and interactions with partially ionized ions

[M. Landreman, A. Stahl, T. Fülöp, CPC 185, 847 (2014)]
[A. Stahl, O. Embréus, G. Papp, M. Landreman, T. Fülöp, NF 56, 112009 (2016)]
Avalanche generation

Cut-off momentum $p_m$:

$$\ln \Lambda \mapsto \ln \Lambda - \ln \sqrt{\frac{\gamma - \gamma_m}{\gamma_m - 1}}.$$ 

$\Rightarrow$ Energy-loss rate independent of $p_m$.

The correction follows from

$$C = C(\theta < \theta_m) + C(\theta > \theta_m)$$

FP approx. Boltzmann

Runaway growth rate [normalized]

$E = 3E_c$ $E = 100E_c$

$0.5$ $1$ $0.5$ $1$

$p_m/p_c$ $p_m/p_c$

Modified ln $\Lambda$ Constant ln $\Lambda$

Field-particle
We can now revisit a classical calculation [R-P, NF 1998]:
The steady state avalanche growth rate

\[ \Gamma = \frac{1}{n_{RE}} \frac{dn_{RE}}{dt} \]
Avalanche generation in a near-threshold electric field

An interesting situation occurs when $E \sim E_c$, as radiation losses become important.

[P. Aleynikov and B. N. Breizman, PRL 114, 155001 (2015)]
Near-threshold electric field

Approximate $\Gamma$ calculated from the avalanche cross-section

$$\Gamma(E) \approx \nu \int_{\gamma_{\text{min}}}^{\gamma_{\text{max}}} \frac{\partial \sigma}{\partial \gamma} d\gamma.$$

Negative growth for small $E$: Reverse knock-ons predicted!

[P. Aleynikov and B. N. Breizman, PRL 114, 155001 (2015)]
Near-threshold electric field

- Significant reverse knock-on however not observed in kinetic simulations
- Runaway decay is described mainly by Fokker-Planck dynamics when $\Gamma \lesssim 0$.

![Graph showing \( \Gamma \) vs. \( E/E_c \)]
Near-threshold electric field

A very important parameter is the effective critical field $E_a : \Gamma(E_a) = 0$, as this quantity sets the current-quench time when $I_{RE} \gg 100 \text{kA}$.

[B. N. Breizman, NF 54, 072002 (2014)]

$\tau_r \propto n_e/B^2$, relative strength of synchrotron energy loss
Bremsstrahlung

Previous work (fusion, lightning, astrophysics) in plasma physics considered only the stopping-power force

\[ F_{\text{brems}}(p) \sim \frac{m_e c}{\tau_{\text{brems}}} \gamma \ln 2 \gamma \]

\[ \gamma = \frac{1}{\sqrt{1 - v^2 / c^2}} \]

However, the average photon energy is large!
A Boltzmann model for radiation losses give significantly different runaway dynamics: Maximum runaway energy significantly underestimated by previous calculations.

[O. Embréus, A. Stahl, T. Fülöp, New Journal of Physics 18, 093023 (2016)]
Summary

The Boltzmann operator has two important applications for runaways:

- **Avalanche runaway generation**
  - Conservative knock-on operator
  - Formally eliminating double counting collisions
  - Describing reverse knock on

- **Bremsstrahlung energy loss**
  - Capturing the finite momenta of emitted photons
  - Accurately describing the energy limit due to radiation losses

Selection of related publications: [Embréus NJP 2016; Stahl PRL 2015; Stahl CPC 2017; Stahl JPCS 2017; Stahl NF 2016; Decker PPCF 2016; Hirvijoki JPP 2015]