# Conservative discontinuous Galerkin discretizations of the 2D incompressible Euler problem

F.L. Waelbroeck[1] C. Michoski[2]

[1] Institute for Fusion Studies, University of Texas at Austin

[2] Institute for Computation and Engineering Science, University of Texas at Austin

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# Outline

# Abstract

Discontinuous Galerkin (DG) methods provide local high-order adaptive numerical schemes for the solution of convection-diffusion problems. They combine the advantages of finite element and finite volume methods. In particular, DG methods automatically ensure the conservation of all first-order invariants provided that single-valued fluxes are prescribed at inter-element boundaries. For the 2D incompressible Euler equation, this implies that the discretized fluxes globally obey Gauss' and Stokes' laws exactly, and that they conserve total vorticity. Combining a continuous Galerkin (CG) solution of Poisson's equation with a central DG flux for the convection term leads to an algorithm that conserves the principal two quadratic invariants, namely the energy and enstrophy. Here, we present a discretization that applies the DG method to Poisson's equation as well as to the vorticity equation while maintaining conservation of the quadratic invariants. **Acknowledgements** This work was

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# Discontinuous Galerkin: A local conservative high-order adaptive method

- The motivation for the discontinuous Galerkin (DG) method is to maintain locality while achieving high-order accuracy on unstructured meshes
  - Unstructured for geometric flexibility
  - High order for accuracy
  - Local for parallel efficiency
  - Automatic element-wise conservation of first-order invariants

| | Complex geometries | High-order accuracy and *hp*-adaptivity | Explicit semi-discrete form | Conservation laws | Elliptic problems |
|---|---|---|---|---|---|
| FDM | × | ✓ | ✓ | ✓ | ✓ |
| FVM | ✓ | × | ✓ | ✓ | (✓) |
| FEM | ✓ | ✓ | × | (✓) | ✓ |
| DG-FEM | ✓ | ✓ | ✓ | ✓ | (✓) |

# Previous conservative DG algorithms

- Liu and Shu [1] have proposed a DG discretization of 2D Euler using a Continuous Galerkin (CG) algorithm to invert Poisson's equation. Their algorithm can conserve enstrophy as well as energy, but allowing some enstrophy decay improves the robustness of the scheme.
- Einkemmer and Wiesenberger [2] have extended the Arakawa method to apply to a DG discretization but this only works for periodic boundary conditions.
- In this poster, we examine the conservation of quadratic invariants for the 2D Euler problem when both Poisson's equation and the vorticity equation are solved using the DG method.

# Governing equations

- The system describing incompressible and inviscid 2D flows is

$$\begin{aligned} \partial_t U &= -\mathbf{e}_z \cdot (\nabla\phi \times \nabla U); &(1)\\ U &= \nabla^2\phi; &(2)\\ \phi &= \phi_b \quad \text{on} \quad \partial\Omega, &(3)\end{aligned}$$

  where $\phi_b$ is given and $\partial\Omega$ is the boundary of the domain $\Omega$.
- This is the archetype of a class of nonlinear fluid problems that includes many multi-scale dynamical systems important in plasma physics.

# Conservation properties of the incompressible 2D Euler model

- The 2D Euler model conserves the following energy:

$$H(\phi) = \frac{1}{2} \int d^2x \, (\nabla \phi)^2. \tag{4}$$

- In addition to conserving $H$, it also conserves the moment and a family of Casimir invariants,

$$\mathbf{M} = \int d^2x \, \mathbf{x} U. \tag{5}$$

$$G(\phi) = \int d^2x \, g(U), \tag{6}$$

- The choice $g(U) = U^2$ corresponds to the enstrophy.

# Weak formulation

- To discretize the equations, we introduce $\mathcal{E} = \nabla\phi$.
- In terms of this auxiliary variable $\mathcal{E}$, Poisson's equation is

$$U = \nabla \cdot \mathcal{E}; \tag{7}$$

$$\phi = \phi_b(\mathbf{x}) \text{ on } \partial\Omega. \tag{8}$$

- The weak formulation of the problem is:

$$\int_K d^2x\, wU_h = \int_{\partial K} ds\, w\hat{\mathcal{E}} \cdot \mathbf{n} - \int_K d^2x\, \mathcal{E}_h \cdot \nabla_h w; \tag{9}$$

$$\int_K d^2x\, \xi \cdot \mathcal{E}_h = \int_{\partial K} ds\, \hat{\phi}\xi \cdot \mathbf{n} - \int_K d^2x\, \phi_h \nabla_h \cdot \xi. \tag{10}$$

Here $\mathbf{n}$ is the outward unit normal to the boundary $\partial K$ of the element, $\hat{\mathcal{E}}$ and $\hat{\phi}$ are the numerical fluxes across the element boundaries

# Gauss' law

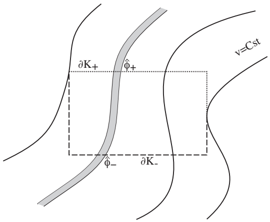
- For $w = 1$ Eq. (9) becomes

$$\int_K d^2x\, U_h = \int_{\partial K} ds\, \hat{\mathcal{E}} \cdot \mathbf{n}, \tag{11}$$

- Note that for a plasma the vorticity is proportional to the polarization charge density, so that the above identity expresses the element-wise applicability of Gauss' law.
- If the numerical flux is conservative, Gauss' law may be extended to macroscopic volumes by summing over elements:

$$\int_\Omega d^2x\, U_h = \int_{\partial\Omega} ds\, \hat{\mathcal{E}} \cdot \mathbf{n}, \tag{12}$$

# Stokes' law




- Taking $\xi = \mathbf{e}_z \times \nabla_h v$ in Eq. (10), we have $\nabla \cdot \xi = 0$.
- Consider a curvlinear coordinate system $(u(x,y), v(x,y))$.
- The length element along lines of constant $v$ is $d\ell = du/\mathbf{e}_\ell \cdot \nabla_h u = |\nabla_h v| \mathcal{J}(v,u)\, du$.
- Changing coordinates in Eq. (10), there follows

$$\int_K dv\, d\ell\, \mathbf{e}_\ell \cdot \boldsymbol{\mathcal{E}}_h = \int_{\partial K^+} dv\, \hat{\phi} - \int_{\partial K^-} dv\, \hat{\phi}.$$

For single-valued $\hat{\phi}$, this can again be extended to macroscopic volumes by summing over $K$s.

# Primal form

- An alternative to solving Eqs. (9)-(10) is to eliminate $\mathcal{E}$, obtaining a variational equation for $\phi$ alone in terms of the primal form $\mathcal{L}_\Omega$ [3]

$$\mathcal{L}_\Omega(\phi_h, w) = -\int_K d^2x\, wU_h, \tag{13}$$

where

$$\mathcal{L}_K(\phi_h, w) = \int_K d^2x\, \nabla_h\phi_h \cdot \nabla_h w - \int_{\partial K} ds\, [(\hat{\phi} - \phi_h)\boldsymbol{n} \cdot \nabla_h w - w\hat{\boldsymbol{\mathcal{E}}} \cdot \boldsymbol{n}] \tag{14}$$

# Interior Penalty method

- The classic IP method corresponds to the choice of fluxes

$$\begin{aligned} \hat{\phi} &= \{\phi_h\} \text{ on } \Gamma_0 \text{ and } \hat{\phi} = 0 \text{ on } \partial\Omega; \quad (15) \\ \hat{\boldsymbol{\mathcal{E}}} &= \{\nabla_h \phi_h\} - \alpha_j([\![\phi_h]\!]) \text{ on } \Gamma; \quad (16) \end{aligned}$$

  where $\alpha(x) = \alpha_j x$ on each edge $e_j$ "penalizes" the discontinuities.
- The corresponding primal form is symmetric,

$$\begin{aligned} \mathcal{L}_\Omega(\phi_h, w) = \int_\Omega d^2x\, \nabla_h \phi_h \cdot \nabla_h w - \int_\Gamma ds\, [\![w]\!] \cdot \alpha([\![\phi_h]\!]) \\ - \int_\Gamma ds\, ([\![\phi_h]\!] \cdot \nabla_h \{w\} + [\![w]\!] \cdot \nabla_h \{\phi_h\}). \quad (17) \end{aligned}$$

# Kinetic energy

- We obtain a numerical kinetic energy by taking the time derivative of Poisson's equation (13) and replacing the test function $w$ by $\phi$:

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}_\Omega(\phi_h, \phi_h) = -\int_\Omega d^2x\, \phi_h \partial_t U_h \tag{18}$$

- The above shows that $\mathcal{L}_\Omega(\phi_h, \phi_h)$ changes at a rate determined by the work done by the convection of vorticity.
- We will show that this is small and can be made to vanish.

# Weak form of the vorticity equation

- The weak form of the vorticity equation is

$$\int_K d^2x\, v\partial_t U_h = \int_{\partial K} ds\, v\widehat{U\mathcal{E}}_h \cdot \mathbf{t} - \int_K d^2x\, U_h(\mathcal{E}_h \times \mathbf{e}_z) \cdot \nabla_h v, \quad (19)$$

  where $v$ is a test function, $\mathbf{t} = \mathbf{e}_z \times \mathbf{n}$ and $\widehat{U\mathcal{E}}_h$ is the numerical flux at element boundaries.

- When using the primal form of Poisson's equation, we must eliminate $\mathcal{E}_h$:

$$\int_K d^2x\, v\partial_t U_h = \int_{\partial K} ds\, (v\widehat{U\mathcal{E}}_h + \hat{\phi} U_h \nabla_h v) \cdot \mathbf{t} + \int_K d^2x\, \phi_h(\nabla U_h \times \mathbf{e}_z) \cdot \nabla_h v, \quad (20)$$

# Energy conservation

- Replacing $w$ by $\phi$ in the weak form of the vorticity equation and integrating by parts yields the following form for the work done by vorticity convection:

$$\int_\Omega d^2x\, \phi_h \partial_t U_h = \int_\Gamma ds\, (\mathbf{n}\cdot [\![\phi_h]\!])\mathbf{t}\cdot (\widehat{U\mathcal{E}}_h - \{U_h \nabla_h \phi_h\}) - \int_{\Gamma^0} ds\, (\hat{\phi} - \{\phi_h\})[\![\mathbf{e}_z \times U_h \nabla_h \phi_h]\!]$$

- Energy is thus conserved by the following choice of flux:

$$\widehat{U\mathcal{E}}_h = \{U_h \nabla_h \phi_h\} \tag{21}$$

# Enstrophy conservation

- Replacing $w$ by $U$ in the weak form of the vorticity equation and integrating yields

$$\frac{1}{2}\frac{d}{dt}\int_K d^2x\, U_h^2 = \int_{\partial K} ds\, [U_h(\mathbf{t}\cdot\widehat{U\mathcal{E}}_h) + \hat{\phi}(\mathbf{t}\cdot\nabla_h U_h^2/2)]$$

- We next integrate the last term by parts. After summing over the elements, we find

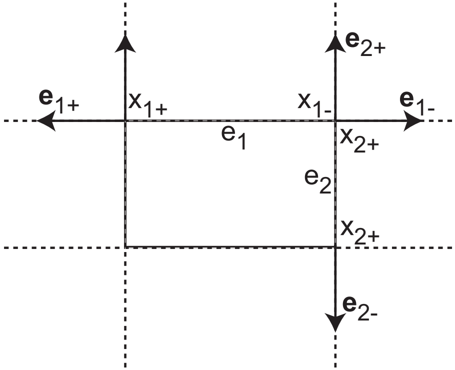
$$\frac{1}{2}\frac{d}{dt}\int_K d^2x\, U_h^2 = \int_\Gamma ds\, (\mathbf{n}\cdot[\![U_h]\!])\, \mathbf{t}\cdot(\widehat{U\mathcal{E}}_h - \{U_h\}\nabla_h\hat{\phi}) \qquad (22)$$

where the $\nabla_h\hat{\phi}$ contains $\delta$-function vertex-contributions from the discontinuities of $\hat{\phi}$.

# Illustration of vertex contributions

- The following choice of flux conserves enstrophy:

$$\widehat{U\mathcal{E}}_h = \{U_h\}\,\nabla_h\hat{\phi} \tag{23}$$



# Summary

- Using DG for Poisson's as well as the vorticity equation, it is possible to conserve enstrophy or vorticity but not both.
- The Poincaré inequality implies that the enstrophy bounds the kinetic energy:

$$\int_\Omega d^2x\,(\nabla\phi)^2 < \int_\Omega d^2x\,U^2.$$

  It follows that conserving enstrophy will ensure stability of both velocity and vorticity.
- Similar techniques can be used to construct DG formulations of the 2D Navier-Stokes equations with the desired stability and conservation properties.
- Numerical implementation is underway.

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