Conservative discontinuous Galerkin discretizations of the 2D incompressible Euler problem

F.L. Waelbroeck\textsuperscript{1}  C. Michoski\textsuperscript{2}

\textsuperscript{1}Institute for Fusion Studies, University of Texas at Austin

\textsuperscript{2}Institute for Computation and Engineering Science, University of Texas at Austin

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Outline

1. Introduction
   - The Discontinuous Galerkin Method
   - The incompressible 2D Euler model

2. Discretization of Poisson’s equation
   - Weak formulation
   - Gauss’ law
   - Stokes’ law
   - Primal form and the kinetic energy

3. Discretization of the vorticity equation
   - Weak form of the vorticity equation
   - Energy conservation
   - Enstrophy conservation

4. Summary
Abstract

Discontinuous Galerkin (DG) methods provide local high-order adaptive numerical schemes for the solution of convection-diffusion problems. They combine the advantages of finite element and finite volume methods. In particular, DG methods automatically ensure the conservation of all first-order invariants provided that single-valued fluxes are prescribed at inter-element boundaries. For the 2D incompressible Euler equation, this implies that the discretized fluxes globally obey Gauss’ and Stokes’ laws exactly, and that they conserve total vorticity. Combining a continuous Galerkin (CG) solution of Poisson’s equation with a central DG flux for the convection term leads to an algorithm that conserves the principal two quadratic invariants, namely the energy and enstrophy. Here, we present a discretization that applies the DG method to Poisson’s equation as well as to the vorticity equation while maintaining conservation of the quadratic invariants. 

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The motivation for the discontinuous Galerkin (DG) method is to maintain locality while achieving high-order accuracy on unstructured meshes:

- Unstructured for geometric flexibility
- High order for accuracy
- Local for parallel efficiency
- Automatic element-wise conservation of first-order invariants

<table>
<thead>
<tr>
<th></th>
<th>Complex geometries</th>
<th>High-order accuracy and $hp$-adaptivity</th>
<th>Explicit semi-discrete form</th>
<th>Conservation laws</th>
<th>Elliptic problems</th>
</tr>
</thead>
<tbody>
<tr>
<td>FDM</td>
<td>$\times$</td>
<td>$\checkmark$</td>
<td>$\checkmark$</td>
<td>$\checkmark$</td>
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<tr>
<td>FVM</td>
<td>$\checkmark$</td>
<td>$\times$</td>
<td>$\checkmark$</td>
<td>$\checkmark$</td>
<td>($\checkmark$)</td>
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<tr>
<td>FEM</td>
<td>$\checkmark$</td>
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<td>$\checkmark$</td>
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<td>DG-FEM</td>
<td>$\checkmark$</td>
<td>$\checkmark$</td>
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</tbody>
</table>
Previous conservative DG algorithms

- Liu and Shu [1] have proposed a DG discretization of 2D Euler using a Continuous Galerkin (CG) algorithm to invert Poisson’s equation. Their algorithm can conserve enstrophy as well as energy, but allowing some enstrophy decay improves the robustness of the scheme.

- Einkemmer and Wiesenberger [2] have extended the Arakawa method to apply to a DG discretization but this only works for periodic boundary conditions.

- In this poster, we examine the conservation of quadratic invariants for the 2D Euler problem when both Poisson’s equation and the vorticity equation are solved using the DG method.
The incompressible 2D Euler model

Governing equations

- The system describing incompressible and inviscid 2D flows is

\[ \partial_t U = -e_z \cdot (\nabla \phi \times \nabla U); \]  
\[ U = \nabla^2 \phi; \]  
\[ \phi = \phi_b \text{ on } \partial \Omega, \]

where \( \phi_b \) is given and \( \partial \Omega \) is the boundary of the domain \( \Omega \).

- This is the archetype of a class of nonlinear fluid problems that includes many multi-scale dynamical systems important in plasma physics.
Conservation properties of the incompressible 2D Euler model

- The 2D Euler model conserves the following energy:

\[ H(\phi) = \frac{1}{2} \int d^2 x (\nabla \phi)^2. \]  

(4)

- In addition to conserving \( H \), it also conserves the moment and a family of Casimir invariants,

\[ M = \int d^2 x x U. \]  

(5)

\[ G(\phi) = \int d^2 x g(U), \]  

(6)

- The choice \( g(U) = U^2 \) corresponds to the enstrophy.
Weak formulation

- To discretize the equations, we introduce $E = \nabla \phi$.
- In terms of this auxiliary variable $E$, Poisson’s equation is
  \[ U = \nabla \cdot E; \]
  \[ \phi = \phi_b(x) \text{ on } \partial \Omega. \]

The weak formulation of the problem is:

\[
\int_K d^2x \, wU_h = \int_{\partial K} ds \, w\hat{E} \cdot n - \int_K d^2x \, E_h \cdot \nabla_h w; \quad (9)
\]
\[
\int_K d^2x \, \xi \cdot E_h = \int_{\partial K} ds \, \hat{\phi} \xi \cdot n - \int_K d^2x \, \phi_h \nabla_h \cdot \xi. \quad (10)
\]

Here $n$ is the outward unit normal to the boundary $\partial K$ of the element, $\hat{E}$ and $\hat{\phi}$ are the numerical fluxes across the element boundaries.
Gauss’ law

- For $w = 1$ Eq. (9) becomes
  \[ \int_K d^2x \ U_h = \int_{\partial K} ds \ \hat{\mathbf{E}} \cdot \mathbf{n}, \]  
  \[ (11) \]

- Note that for a plasma the vorticity is proportional to the polarization charge density, so that the above identity expresses the element-wise applicability of Gauss’ law.

- If the numerical flux is conservative, Gauss’ law may be extended to macroscopic volumes by summing over elements:
  \[ \int_{\Omega} d^2x \ U_h = \int_{\partial \Omega} ds \ \hat{\mathbf{E}} \cdot \mathbf{n}, \]  
  \[ (12) \]
Stokes’ law

- Taking \( \xi = e_z \times \nabla_h v \) in Eq. (10), we have \( \nabla \cdot \xi = 0 \).
- Consider a curvilinear coordinate system \((u(x, y), v(x, y))\).
- The length element along lines of constant \( v \) is
  \[
  d\ell = \frac{du}{e_\ell \cdot \nabla_h u} = |\nabla_h v| J(v, u) du.
  \]
- Changing coordinates in Eq. (10), there follows
  \[
  \int_K dv \; d\ell \; e_\ell \cdot \mathcal{E}_h = \int_{\partial K^+} dv \hat{\phi} - \int_{\partial K^-} dv \hat{\phi}.
  \]

For single-valued \( \hat{\phi} \), this can again be extended to macroscopic volumes by summing over \( K \)s.
An alternative to solving Eqs. (9)-(10) is to eliminate $\mathcal{E}$, obtaining a variational equation for $\phi$ alone in terms of the primal form $\mathcal{L}_\Omega$ [3]

$$\mathcal{L}_\Omega(\phi_h, w) = -\int_K d^2 x \, w U_h,$$

where

$$\mathcal{L}_K(\phi_h, w) = \int_K d^2 x \, \nabla_h \phi_h \cdot \nabla_h w$$

$$- \int_{\partial K} ds \left[ (\hat{\phi} - \phi_h) n \cdot \nabla_h w - w \hat{\mathcal{E}} \cdot n \right]$$
The classic IP method corresponds to the choice of fluxes

\[
\hat{\phi} = \{\phi_h\} \text{ on } \Gamma_0 \text{ and } \hat{\phi} = 0 \text{ on } \partial\Omega; \\
\hat{E} = \{\nabla_h \phi_h\} - \alpha_j([\phi_h]) \text{ on } \Gamma;
\]

(15) \quad (16)

where \(\alpha(x) = \alpha_j x\) on each edge \(e_j\) “penalizes” the discontinuities.

The corresponding primal form is symmetric,

\[
\mathcal{L}_\Omega(\phi_h, w) = \int_\Omega d^2 x \nabla_h \phi_h \cdot \nabla_h w - \int_\Gamma ds \left[ [w] \cdot \alpha([\phi_h]) \right] \\
- \int_\Gamma ds (\{\phi_h\} \cdot \nabla_h \{w\} + [w] \cdot \nabla_h \{\phi_h\}).
\]

(17)
We obtain a numerical kinetic energy by taking the time derivative of Poisson’s equation (13) and replacing the test function \( w \) by \( \phi \):

\[
\frac{1}{2} \frac{d}{dt} \mathcal{L}_\Omega(\phi_h, \phi_h) = - \int_\Omega d^2 x \, \phi_h \partial_t U_h \tag{18}
\]

The above shows that \( \mathcal{L}_\Omega(\phi_h, \phi_h) \) changes at a rate determined by the work done by the convection of vorticity.

We will show that this is small and can be made to vanish.
The weak form of the vorticity equation is

$$\int_K d^2 x \nu \partial_t U_h = \int_{\partial K} ds v \hat{U} \mathcal{E}_h \cdot t - \int_K d^2 x U_h (\mathcal{E}_h \times \mathbf{e}_z) \cdot \nabla_h v,$$  \hspace{1cm} (19)

where $\nu$ is a test function, $t = \mathbf{e}_z \times \mathbf{n}$ and $\hat{U} \mathcal{E}_h$ is the numerical flux at element boundaries.

When using the primal form of Poisson’s equation, we must eliminate $\mathcal{E}_h$:

$$\int_K d^2 x \nu \partial_t U_h = \int_{\partial K} ds (v \hat{U} \mathcal{E}_h + \hat{\phi} U_h \nabla_h v) \cdot t$$

$$+ \int_K d^2 x \phi_h (\nabla U_h \times \mathbf{e}_z) \cdot \nabla_h v,$$  \hspace{1cm} (20)
Replacing \( w \) by \( \phi \) in the weak form of the vorticity equation and integrating by parts yields the following form for the work done by vorticity convection:

\[
\int_{\Omega} d^2 x \; \phi_h \partial_t U_h = \int_{\Gamma} ds (n \cdot \llbracket \phi_h \rrbracket) t \cdot (\widehat{U\varepsilon}_h - \{U_h \nabla_h \phi_h\})
- \int_{\Gamma_0} ds (\hat{\phi} - \{\phi_h\}) \llbracket e_z \times U_h \nabla_h \phi_h \rrbracket
\]

Energy is thus conserved by the following choice of flux:

\[
\widehat{U\varepsilon}_h = \{U_h \nabla_h \phi_h\} \quad (21)
\]
Enstrophy conservation

- Replacing $w$ by $U$ in the weak form of the vorticity equation and integrating yields

$$
\frac{1}{2} \frac{d}{dt} \int_K d^2 x \ U_h^2 = \int_{\partial K} ds \left[ U_h (t \cdot \overline{UE}_h) + \hat{\phi} (t \cdot \nabla h U_h^2 / 2) \right]
$$

- We next integrate the last term by parts. After summing over the elements, we find

$$
\frac{1}{2} \frac{d}{dt} \int_K d^2 x \ U_h^2 = \int_{\Gamma} ds \left( n \cdot [ U_h ] \right) t \cdot ( \overline{UE}_h - \{ U_h \} \nabla h \hat{\phi} ) \tag{22}
$$

where the $\nabla h \hat{\phi}$ contains $\delta$-function vertex-contributions from the discontinuities of $\hat{\phi}$. 
Illustration of vertex contributions

The following choice of flux conserves enstrophy:

\[
\widehat{U\mathcal{E}}_h = \{U_h\} \nabla_h \widehat{\phi}
\]  (23)
Using DG for Poisson’s as well as the vorticity equation, it is possible to conserve enstrophy or vorticity but not both. The Poincaré inequality implies that the enstrophy bounds the kinetic energy:

\[
\int_{\Omega} d^2 x (\nabla \phi)^2 < \int_{\Omega} d^2 x U^2.
\]

It follows that conserving enstrophy will ensure stability of both velocity and vorticity.

Similar techniques can be used to construct DG formulations of the 2D Navier-Stokes equations with the desired stability and conservation properties.

Numerical implementation is underway.
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