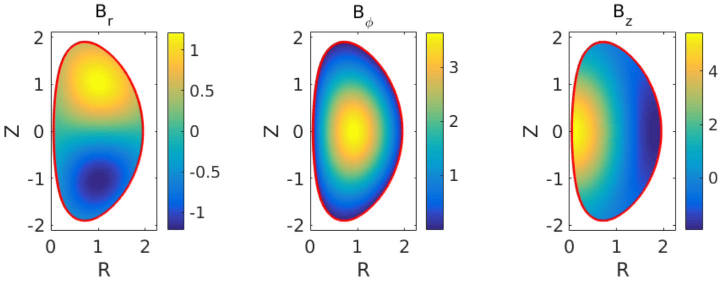
A fast, high order accurate solver for Taylor states in general toroidal geometries



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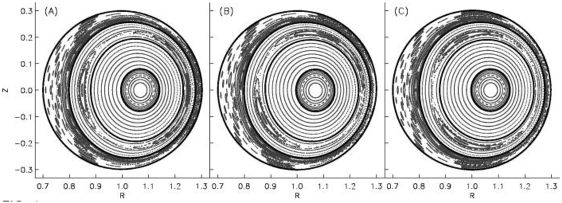
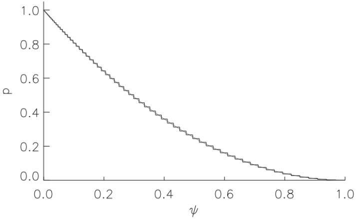
MOTIVATION

- 3D equilibrium calculations with SPEC[1]





- Plasma domain divided into regions which are in relaxed Taylor states, separated by ideal MHD interfaces
- Magnetic field for a Taylor state given by

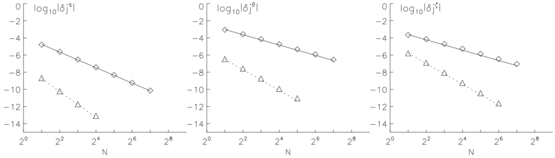
$$\nabla \times \mathbf{B} = \mu \mathbf{B}$$

with $\mu = cst$

[1]S.R. Hudson, R.L. Dewar, G. Dennis, M.J. Hole, M. McGann, G. von Nessi, and S. Lazerson, "Computation of multi-region relaxed magnetohydrodynamic equilibria", *Phys. Plasmas* **19**, 112502 (2012)

MOTIVATION - SOLVER FOR TAYLOR RELAXED STATE

- SPEC solves $\nabla \times \mathbf{B} = \mu \mathbf{B}$ with a Fourier discretization in angle, and finite elements in the radial direction



$$\delta \mathbf{j} = \mathbf{j} - \mu \mathbf{B}$$

- Idea: Use integral equation representation to solve for the Beltrami field in each region
  - $\nabla \times \mathbf{B} - \mu \mathbf{B} \equiv 0$ identically satisfied
  - Same high order convergence for $\mathbf{B}$ and for $\mathbf{J}$
  - Low memory requirements (only the boundary of the domain has to be discretized)
  - Fast algorithms available ($O(N \log N)$ for N degrees of freedom)

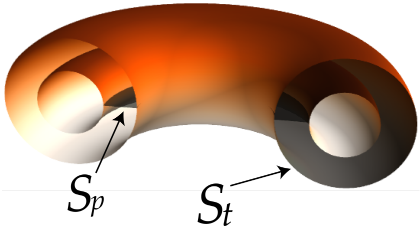
Well-posed problem

- For a given μ, solve for $\mathbf{B}$ solution of

$$\nabla \times \mathbf{B} = \mu \mathbf{B} \qquad \text{in } \Omega$$

$$\mathbf{n} \cdot \mathbf{B} = 0 \qquad \text{on } \Gamma \equiv \partial\Omega$$

$$\int_{S_t} \mathbf{B} \cdot d\mathbf{S} = \Phi^{tor} \qquad \text{and} \qquad \int_{S_p} \mathbf{B} \cdot d\mathbf{S} = \Phi^{pol}$$



- Note: if Ω has genus 1, one flux condition is sufficient

Spinors as solutions to Beltrami equation

- Time-harmonic Maxwell's equations in a region free of charge and current:

$$\nabla \times \mathbf{E} = i\omega \mathbf{H} \qquad \nabla \times \mathbf{H} = -i\omega \mathbf{E}$$

- Spinor $\mathbf{S}_+ = \mathbf{E} + i\mathbf{H}$ satisfies

$$\nabla \times \mathbf{S}_+ = i\omega \mathbf{H} - i(i\omega \mathbf{E}) = \omega \mathbf{S}_+$$

- Choose Lorenz gauge for $\mathbf{E}$ and $\mathbf{H}$

$$\mathbf{H} = \nabla \times \mathbf{A} \qquad \mathbf{E} = i\omega \mathbf{A} - \nabla u \qquad \nabla \cdot \mathbf{A} = i\omega u$$

  to represent Beltrami field $\mathbf{B}$ as

$$\mathbf{B} = i\omega \mathbf{A} - \nabla u + i\nabla \times \mathbf{A}$$

- Replace ω with μ for Beltrami problem of interest
- Need a representation for $\mathbf{A}$ and u

Generalized Debye representation of the potentials [2]

- **A** and u are written as

$$\mathbf{A}(\mathbf{x}) = \int_\Gamma \frac{e^{i\mu|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}\mathbf{m}(\mathbf{x}')dA' \qquad u(\mathbf{x}) = \int_\Gamma \frac{e^{i\mu|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}\sigma(\mathbf{x}')dA'$$

- **m** and σ are related through

$$\mathbf{m} = i\mu\left(\nabla_\Gamma\Delta_\Gamma^{-1}\sigma - i\,\mathbf{n}\times\nabla_\Gamma\Delta_\Gamma^{-1}\sigma\right) + \alpha\,\mathbf{m}_H.$$

∇_Γ is the surface gradient operator
Δ_Γ^{-1} is the inverse of the surface Laplacian along Γ restricted to the class of *mean-zero functions*
$\mathbf{m}_H$ is a tangential harmonic vector field satisfying

$$\nabla_\Gamma\cdot\mathbf{m}_H = 0, \qquad \nabla_\Gamma\cdot\mathbf{n}\times\mathbf{m}_H = 0 \qquad \mathbf{n}\times\mathbf{m}_H = -i\,\mathbf{m}_H.$$

α: complex number determined by B.C.

[2] C.L. Epstein, L. Greengard, and M. ONeil. "Debye Sources, Beltrami Fields, and a Complex Structure on Maxwell Fields", *Comm. Pure Appl. Math.* **68**(12):22372280, 2016.

INTEGRAL EQUATION FOR σ AND α

- Applying $\mathbf{B} \cdot \mathbf{n} = 0$ and the flux condition leads to the following integral equations for σ and α:

$$\frac{\sigma}{2} - \mathbf{n} \cdot \nabla \int_\Gamma \frac{e^{i\mu|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}\sigma(\mathbf{x}')dS'$$
$$+ i\mu\mathbf{n} \cdot \int_\Gamma \frac{e^{i\mu|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}\mathbf{m}dS' + i\mathbf{n} \cdot \nabla \times \int_\Gamma \frac{e^{i\mu|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}\mathbf{m}dS' = 0$$

$$\frac{1}{\mu}\int_{\partial S_t} \mathbf{B} \cdot d\mathbf{l} = \Phi^{tor}$$

- Well-conditioned, second kind integral equation
- Similar formulation (with more terms) for toroidal *shells*

NUMERICS

- Use 16^{th} order hybrid Gauss-trapezoidal rule to evaluate singular integrals
- Use Fourier spectral differentiation matrix to evaluate ∇_Γ
- Compute Δ_Γ^{-1} by solving

$$(\Delta_\Gamma + \int_\Gamma dS)\omega = f$$

  Invertible equation, and ω satisfies $\Delta_\Gamma \omega = f$ and $\int_\Gamma \omega dS = 0$
- Major simplifications for axisymmetric equilibria:
  1. There is a closed form formula for the harmonic surface vector field $\mathbf{m}_H$
  2. High order accuracy achieved with few unknowns $\Rightarrow$ dense linear algebra solvers fast

Testing the solver: constructing exact Taylor states [3]

- View Taylor state as Grad-Shafranov equilibrium

$$\Delta^* \psi = -\mu^2 \psi \qquad \text{in } \Omega, \qquad \psi = 0 \ \text{ on } \Gamma$$

- A general solution is

$$\psi(r, z, c_1, c_2, c_3, c_4, c_5, c_6, \lambda) = \psi_0 + c_1\psi_1 + c_2\psi_2 + c_3\psi_3 + c_4\psi_4 + c_5\psi_5$$

$$\psi_0 = rJ_1(\mu r)\ ,\ \psi_1 = rY_1(\mu r)\ ,\ \psi_2 = rJ_1\left(\sqrt{\mu^2 - c_6^2}\, r\right)\cos(c_6 z)$$

$$\psi_3 = rY_1\left(\sqrt{\mu^2 - c_6^2}\, r\right)\cos(c_6 z)\ ,\ \psi_4 = \cos\left(\mu\sqrt{r^2 + z^2}\right)$$

$$\psi_5 = \cos(\mu z)$$

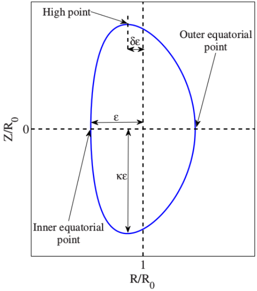
- The toroidal flux is then given by $\Phi^{tor} = \mu \iint_\Omega \frac{\psi}{r} drdz$
- For Taylor states with X-points, need 5 more terms and 5 more c_i

[3] A.J. Cerfon and M. O'Neil, "Exact axisymmetric Taylor states for shaped plasmas", *Phys. Plasmas* **21**, 064501 (2014)

TESTING THE SOLVER: CONSTRUCTING EXACT TAYLOR STATES

- Treat μ as unknown along with the 6 c_i
- Solve for the unknowns by imposing 7 conditions on $\psi = 0$ curve

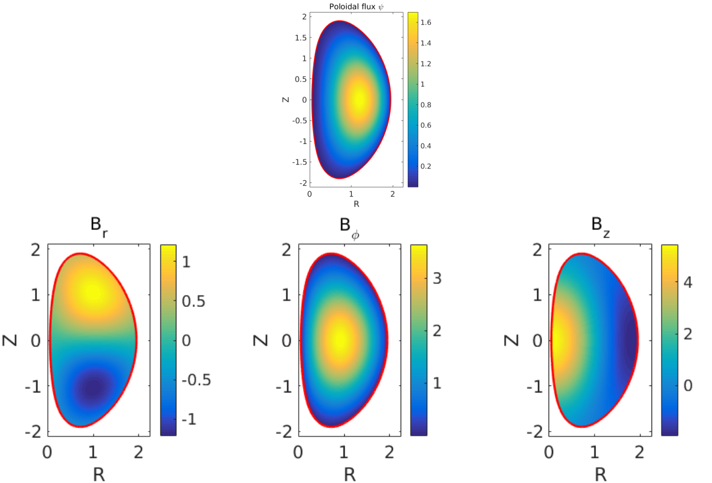
$$
\begin{cases}
\psi(1+\epsilon, 0, C) = 0 \\
\psi(1-\epsilon, 0, C) = 0 \\
\psi(1-\delta\epsilon, -\kappa\epsilon, C) = 0 \\
\psi_r(1-\delta\epsilon, -\kappa\epsilon, C) = 0 \\
\psi_{zz}(1+\epsilon, 0, C) + N_1\psi_r(1+\epsilon, 0, C) = 0 \\
\psi_{zz}(1-\epsilon, 0, C) + N_2\psi_r(1-\epsilon, 0, C) = 0 \\
\psi_{rr}(1-\delta\epsilon, -\kappa\epsilon, C) + N_3\psi_z(1-\delta\epsilon, -\kappa\epsilon, C) = 0
\end{cases}
$$



N_1, N_2, N_3: curvatures at three points $(1+\epsilon, 0)$, $(1-\epsilon, 0)$, $(1-\delta\epsilon, \kappa\epsilon)$

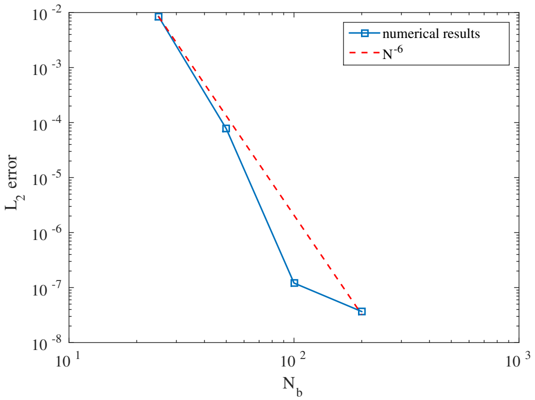
Comparison with exact Taylor state

- Challenging very low aspect ratio, high elongation Taylor state:
  $\epsilon = 0.95$, $\kappa = 2$, $\delta = 0.3$
  $\mu = 2.281569790667846$



RESULTS

N_b	$\|\|\mathbf{B}_{ex} - \mathbf{B}_{num}\|\|_2$	$T_{build}(s)$	$T_{solve}(s)$
25	8.49E-03	1.4E-01	4.8E-05
50	7.74E-05	3.8E-01	1.2E-04
100	1.21E-07	1.3E+00	5.5E-04
200	3.67E-08	6.0E+00	3.5E-03



Summary - Future work

- A generalized Debye representation of the magnetic field is a natural formulation for magnetic fields in Taylor states
- The formulation leads to a well-conditioned integral equation of the second kind
- The associated numerical solver is high order accurate and has low memory requirements

- Future work:
  - (Near term) More complete convergence tests for toroidal and toroidal shell regions
  - (Longer term, though already ongoing): apply method to nonaxisymmetric Taylor states
    1. Numerical computation of harmonic surface vector fields (Lise-Marie Imbert-Gérard)
    2. Implement fast direct solvers or FMM methods for dense linear system