# The effect of sheared toroidal rotation on pressure driven islands in toroidal plasmas

C. C. Hegna
University of Wisconsin
Madison, WI  53706

2016 International Sherwood Fusion Theory Meeting
Madison, WI
April 2, 2016

# Theses

- The impact of sheared toroidal rotation on pressure driven magnetic island widths in tokamak plasmas is assessed

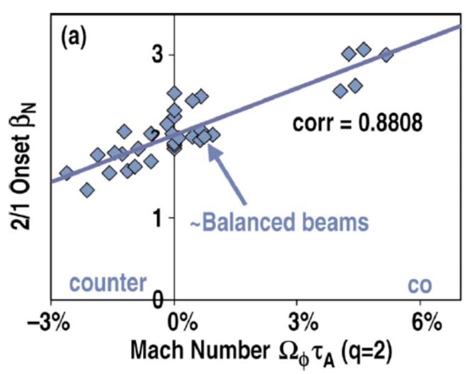
- Sheared flow enhances the stabilizing effect of favorable average-curvature and pressure gradient (Glasser term). No impact on NTM drive → sheared flow tends to have a net reduction on pressure driven island in tokamaks

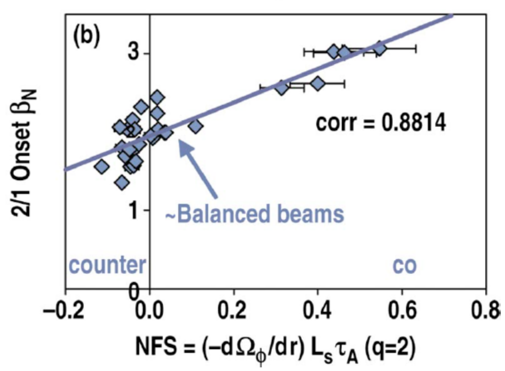
- The effect of the sheared flow is enhanced in toroidal plasmas over that predicted in cylindrical plasmas by a Pfirsch-Schlüter-like correction ~ $1 + 2q^2$

# Empirically, it's known that sheared flow alters onset conditions for NTMs and the saturated island widths

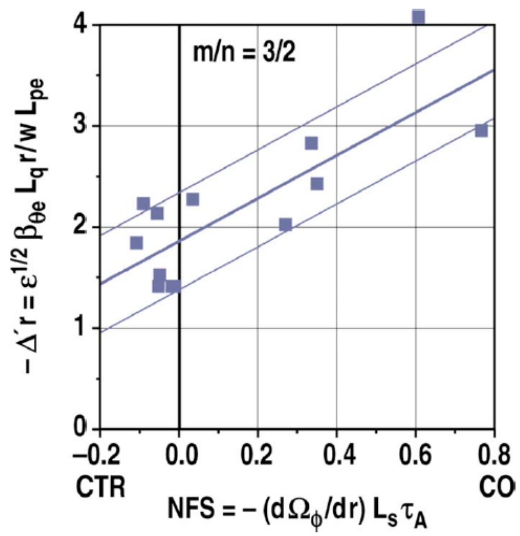
- A variety of studies have shown sensitivity to flow shear physics
  LaHaye et al, PoP 056110 (2010), Buttery et al, PoP 056115 (2008), Gerhardt et al, NF 49, 032003 (2009), …

Onset $\beta_N$ for NTMs increases with toroidal flow shear




Saturated island width decreases with flow shear



# Large number of studies have been dedicated to shear flow stabilization of tearing modes

- A number of mechanisms for shear flow to modify tearing modes:
- From Linear theory
  - Modification of $\Delta'$ --- generally stabilizing (Chen and Morrison, ‘90, Chandra et al ’05, Sen et al ‘13 …)
  - Differential rotation between geometrically coupled rational surfaces, resonant field errors and/ or resistive walls --- generally stabilizing (Fitzpatrick ‘93, Chandra ‘07 …)
- Nonlinear physics
  - Flow modifications to nonlinear island region currents --- polarization currents (Fitzpatrick and Waelbroeck ‘09), …
  - Flow modifications pressure drives in island region

# Toroidal MHD equilibria are considered with sheared toroidal flow

- MHD equilibrium with sheared toroidal flow

$$\vec{B}_0 = I(\psi)\nabla\zeta + \nabla\zeta \times \nabla\psi$$

$$\vec{v}_0 = \Omega(\psi) R^2 \nabla\zeta$$

  – Grad-Shafranov with toroidal flow

$$\Delta^*\psi = -I\frac{dI}{d\psi} - \mu_o R^2 \frac{\partial}{\partial\psi} P_o(\psi) e^{\rho_o R^2 \Omega^2 / 2p_o}$$

    - For this work, we will work in the weak flow, but strong flow shear regime

$$\Omega(\psi) = \cancelto{0}{\Omega(\psi_0)} + \frac{d\Omega}{d\psi}(\psi - \psi_o) + \ldots \approx \frac{d\Omega}{d\psi}(\psi - \psi_o)$$

    - Motivated by observations that flow shear is more important than flow in NTM physics

# Sheared flow effects modify Mercier stability indices and asymptotic matching

- Asymptotic matching is determined by Mercier indices

$$A_{||} \rightarrow A_l \,|\, x \,|^{\alpha_l} + A_s \,|\, x \,|^{\alpha_s} \qquad \Delta' = \frac{A_{s+}}{A_{l+}} + \frac{A_{s-}}{A_{l-}}$$

  - In the presence of shear flow, Mercier indices are modified. In the small $\gamma\mu_o p/B^2 << 1$ limit (Chu, '98)

$$\alpha_{l,s} = \frac{1}{2} \mp \sqrt{-D_{IM_A}} = \frac{1}{2} \mp \sqrt{\frac{1}{4} - \frac{E+F+H}{1-M_A^2}}$$

    - E, F, H = conventional measure of interchange stability (Glasser et al PF '75)
    - Sheared flow measured by $M_A$

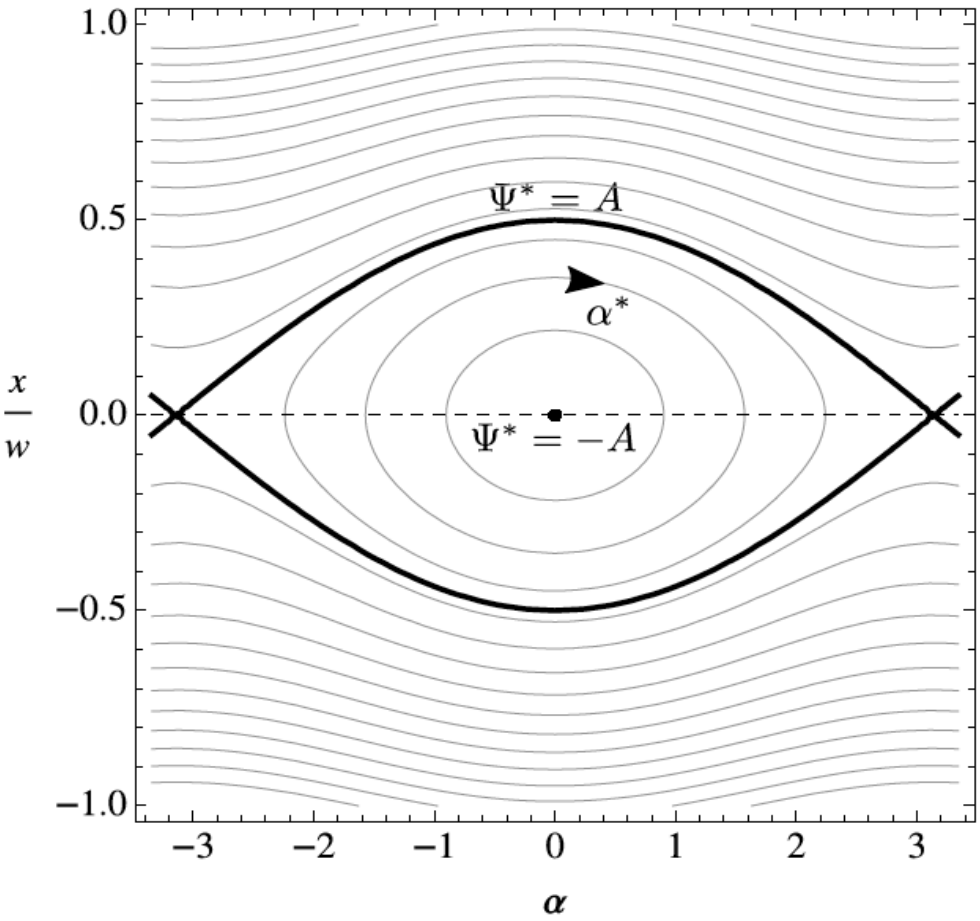
$$M_A = \frac{\hat{V}' \sqrt{\rho_o \mu_o M}}{dq/d\psi} \frac{d\Omega}{d\psi} \cong \frac{1}{V_A} \frac{q}{dq/dr} \frac{dV_\xi}{dr} \sqrt{1+2q^2}$$



*Hegna, Sherwood, 4/2/16*
*University of Wisconsin-Madison*

# Nonlinear island dynamics are considered using a Rutherford theory-like approach

- Flux surfaces labeled by helical flux $\Psi^*$




$$\vec{B} = \vec{B}_0 + \vec{B}_1 = \nabla\alpha \times \nabla\Psi^* + \nabla\psi \times \nabla\chi$$

$$\bar{\Psi}^* = \int d\psi (q - q_o) - A\cos(n_o\alpha)$$

$$\cong q_o' \frac{x^2}{2} - A\cos(n_o\alpha)$$

$$\alpha = \zeta - q_o\theta \qquad q_o = \frac{m_o}{n_o}$$

$$\chi = \theta$$

$$x = \psi - \psi_o$$

$$w = 4\sqrt{\frac{A}{|q_o'|}}$$


*Hegna, Sherwood, 4/2/16*
*University of Wisconsin-Madison*

# Nonlinear neoclassical-MHD equations employed

- Island Grad-Shafranov-like equation determined from ideal MHD

$$\rho\vec{v}\cdot\nabla\vec{v} = \vec{J}\times\vec{B} - \nabla p,$$
$$-\nabla\Phi + \vec{v}\times\vec{B} = 0,$$
$$\nabla\cdot(\rho\vec{v}) = 0,$$
$$\vec{v}\cdot\nabla p + \gamma p\nabla\cdot\vec{v} = 0,$$

  - Rutherford-style matched asymptotic solutions
    - $J_{||}/B$ in the island region calculated self-consistently with flow modified profiles
    - Asymptotic matching to exterior data
    - Small island approximation
      - Arbitrary p', axisymmetric shaping
      - Small beta approximation when the going gets tough

# Properties of flow and pressure profiles deduced from leading order equations

From lowest order Ohm's law, pressure equation and parallel momentum balance, pressure and flow profiles satisfy:

$$\vec{B} \cdot \nabla \Phi = 0,$$

$$\vec{v} \cdot \nabla p = 0,$$

$$\nabla \cdot \vec{v} = 0$$

- Equilibration of pressure and electrostatic potential on helical flux surfaces, Pfirsch-Schlüter plasma flows

$$\Phi = \Phi(\bar{\Psi}^*)$$

$$p = p(\bar{\Psi}^*)$$

$$\frac{v_\parallel}{B} = \frac{< v_\parallel B >_x}{< B^2 >_x} - \lambda \frac{\partial \Phi}{\partial x}$$

– $\lambda$ = Pfirsch-Schlüter coefficient

$$\vec{B} \cdot \nabla \lambda = -\nabla \cdot \frac{\vec{B} \times \nabla \psi}{B^2} \qquad \lambda = \frac{1}{B^2} - \frac{1}{< B^2 >_X}$$



*Hegna, Sherwood, 4/2/16*
*University of Wisconsin-Madison*

# Quasineutrality condition predicts helical Pfirsch-Schlüter and polarization currents

- Quasineutrality condition

$$\vec{B}\cdot\nabla\frac{J_{\parallel}}{B} = -\nabla\cdot\frac{\overline{B}\times(\nabla p + \rho\vec{v}\cdot\nabla\vec{v})}{B^2}$$

  - Parallel current profile has a number of terms

$$\frac{J_{\parallel}}{B} = -\lambda\frac{\partial p}{\partial x} + \frac{\hat{V}'}{\mu_o < B^2/g^{\psi\psi} >_X}\frac{q_o'}{p_o'}\frac{dp}{d\overline{\Psi}*}[-q_o' x(E+F) + H\frac{\partial A}{\partial x}] - \frac{\hat{V}'^2\rho M}{< B^2/g^{\psi\psi} >_X}\frac{d\Phi}{d\overline{\Psi}^*}\frac{\partial^2\Phi}{\partial x^2} + F(\overline{\Psi}^*)$$

Toroidal Pfirsch-Schlüter currents

Helical Pfirsch-Schlüter currents – driven by Interchange physics

Polarization currents

Function, to be determined by transport physics

- Flow profile factor augmented in toroidal geometry by the quantity M → M ~ 1 + 2q$^2$ in large aspect ratio limit

$$M = < \frac{B^2}{g^{\psi\psi}} >_x (< \frac{g^{\psi\psi}}{B^2} >_x + < \lambda^2 B^2 >_x)$$

# Ampere's law used to derive self-consistency relation

- Island Grad-Shafranov expression derived from Ampere's law

$$(1-M_a^2)\frac{\partial^2 A}{\partial x^2} = \frac{q_o^{'}}{p_o^{'}}\frac{dp}{d\bar{\Psi}*}[q_o^{'}x(E+F)-H\frac{\partial A}{\partial x}]$$

$$+M_a\frac{dM_a}{d\bar{\Psi}^*}(\partial\bar{\Psi}/\partial x)^2 + H\frac{q_o^{'}}{p_o^{'}}(\frac{\partial p}{\partial x}-p_o^{'}) + F(\bar{\Psi}^*)$$

  – Dimensionless profile function $M_a$ denotes normalized island flow profile

$$M_a(\bar{\Psi}^*) \equiv \hat{V}'\sqrt{\rho\mu_o M}\frac{d\Phi}{d\bar{\Psi}^*}$$

    - Profile function asymptotes to $M_A$ at large |x|

$$\lim_{|x|\to\infty} M_a(\bar{\Psi}^*) = M_A$$

# Island Grad-Shafranov Equation determined to within three free functions of $\Psi^*$

- Three profile functions required:
  - $p(\Psi^*)$ --- pressure
  - $M_a(\Psi^*)$ --- flow
  - $F(\Psi^*)$ --- current
- Profile functions constrained by transport physics
  - Pressure determined by constancy of net pressure flux and Fick's law form $\Gamma_p = -D_p \nabla p$
  - Flow profile determined from no net cross-field radial current and Fick's law form for viscous stress $\vec{\vec{\pi}} = -\rho\mu\nabla\vec{v}$
    - $M_a(\Psi^*) = M_A$
  - Current profile --- resistive-neoclassical Ohm's law


*Hegna, Sherwood, 4/2/16*
*University of Wisconsin-Madison*

# Neoclassical Ohm’s law allows for the inclusion of bootstrap current contributions

- Neoclassical Ohm’s law

$$< \vec{E}\cdot\vec{B} >_{*} = - < \vec{B}\cdot\frac{\partial \vec{A}}{\partial t} >_{*} = \eta < J_{\|}B >_{*} - \frac{< \vec{B}\cdot\nabla\cdot\Pi_{e} >_{*}}{n_{e}e}$$

- Provides transport constraint for current profile

$$\frac{\partial^{2}\bar{A}}{\partial x^{2}} = \frac{q_{o}^{'}}{p_{o}^{'}}\frac{dp}{d\bar{\Psi}^{*}}[\frac{E+F}{1-M_{a}^{2}}q_{o}^{'}(x - < x >_{*}) + \frac{H}{1-M_{a}^{2}}(< \frac{\partial \bar{A}}{\partial x} >_{*} - \frac{\partial \bar{A}}{\partial x}) + \frac{H}{1-M_{a}^{2}}\frac{q_{o}^{'}}{p_{o}^{'}}[\frac{\partial \delta p}{\partial x} - M_{a}^{2} < \frac{\partial \delta p}{\partial x} >_{*}]$$

$$+ \frac{M_{a}}{1-M_{a}^{2}}\frac{dM_{a}}{d\bar{\Psi}^{*}}[(\frac{\partial \bar{\Psi}^{*}}{\partial x})^{2} - < (\frac{\partial \bar{\Psi}^{*}}{\partial x})^{2} >_{*}] + \frac{\mu_{o}}{\eta_{nc}}\frac{< B^{2}/g^{\psi\psi} >_{x}}{< B^{2} >_{x}} < \frac{\partial \bar{A}}{\partial t} >_{*} + \frac{\mu_{e}}{\mu_{e}+\nu_{e}} < \frac{\partial \delta p}{\partial x} >_{x} \frac{\mu_{o} I \hat{V}' < B^{2}/g^{\psi\psi} >_{x}}{< B^{2} > x}$$

  - Resistive interchange contributions
  - Polarization effects
  - Bootstrap current drive ~ NTM drive
  - Inductive electric field ~ (dw/dt)

Amplified by sheared toroidal flow by factor $(1 - M_A^2)^{-1}$ (arrows point to "Resistive interchange contributions" and "Polarization effects")


*Hegna, Sherwood, 4/2/16*
*University of Wisconsin-Madison*

# Island Grad-Shafranov equation is to be matched asymptotically to exterior region

- Asymptotics of island region solution
  - At large |x|, inner layer solution

$$\frac{\partial^2 A}{\partial x^2} + \frac{1}{x^2}\frac{E+F+H}{1-M_A^2}A = 0$$

$$A_{\|} \rightarrow A_l \,|\, x \,|^{\alpha_l} + A_s \,|\, x \,|^{\alpha_s} \qquad \Delta' = \frac{A_{s+}}{A_{l+}} + \frac{A_{s-}}{A_{l-}}$$

$$\alpha_{ls} = \frac{1}{2} \mp \sqrt{\frac{1}{4} - \frac{E+F+H}{1-M_A^2}}$$

  - Matching to exterior region

$$\lim_{x'\rightarrow\infty} |\, x \,|^{\alpha_l} \int_{-x'}^{x'} dx \oint \frac{d\alpha}{\pi}\cos(n\alpha)\frac{\partial^2 A}{\partial x^2} = \Delta' A_l \sqrt{-4D_{IM_A}}$$


*Hegna, Sherwood, 4/2/16*
*University of Wisconsin-Madison*

# Modified Rutherford equation is derived that includes effect of sheared flow on pressure drives

- Modified Rutherford equation

$$k_0 \frac{\mu_o}{\eta_{nc}} \frac{dw}{dt} = w^{-2\alpha_l} \Delta^* + \frac{k_1}{w} (D_{nc} + D_{RM_A})$$

$$\Delta^* = \Delta' 2^{2\alpha_l} \sqrt{-4 D_{IM_A}}$$

$$D_{nc} = -\frac{\mu_e}{\mu_e + \nu_e} \frac{\mu_o p_o' q_o}{q_o'} \frac{\overline{R}^2 < B^2 / g^{\psi\psi} >}{< B^2 >}$$

NTM drive --- destabilizing in tokamaks

$$D_{RM_A} = \frac{E + F + H^2 + M_A^2 (H - H^2)}{(1 - M_A^2)(\alpha_s - H)}$$

Resistive interchange contribution --- generally stabilizing

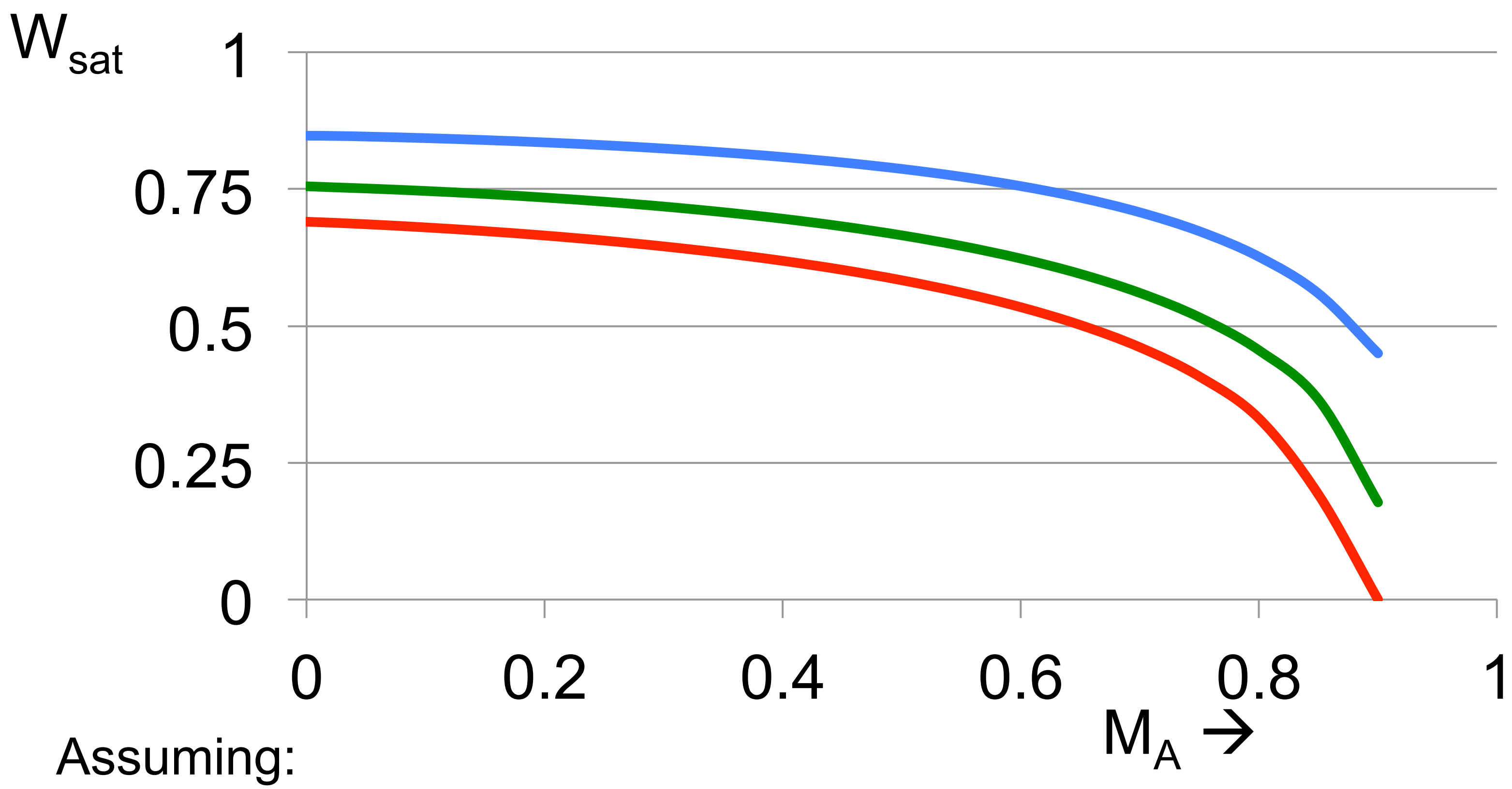
  - Flow shear amplifies the stabilizing effect of good averaged curvature -- generally $(E + F + H^2) < 0$ in tokamak equilibria
  - Modifies $\Delta'$ contribution

$$(w_{sat})^{1-2\alpha} = k_1 (D_{NC} + D_{RMA}) / (-\Delta' (-4 D_{IMA})^{0.5})$$



*Hegna, Sherwood, 4/2/16*
*University of Wisconsin-Madison*

# Sheared flow produces modest corrections to saturated island width




Assuming:

$D_{nc} = 1$

$E + F = -0.05, -0.1, -0.15$

$H = 0.025, 0.05, 0.075$



*Hegna, Sherwood, 4/2/16*
*University of Wisconsin-Madison*

# Conclusions

- The effects of sheared toroidal flow are included in calculations of saturated pressure driven magnetic islands using a neoclassical-resistive MHD model.
- Sheared toroidal flow enhances the stabilizing effects of favorable pressure gradient/average curvature by the factor ~ $1/(1 - M_A^2)$
- The effect of sheared flow is enhanced over that predicted in cylindrical theory by a factor of order $1 + 2q^2$ due to a Pfirsch-Schlüter-like correction
- The sheared flow correction should also enhance ion polarization currents responses as well --- an effect to be addressed in future work.