# Variational approaches to the guiding-center Vlasov-Maxwell equations*

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The Vlasov equation for guiding center motion is coupled to a self-evolving electromagnetic field to produce the guiding-center Vlasov-Maxwell equations. In the same spirit as [3, 4], we follow a variational approach in order to capture consistent nonlinear effects and ensure energy and momentum balance. In particular, we use several variational approaches to show how the magnetization term in Ampère's Law retains a moving electric-dipole correction, whose role has not not always been given proper attention and still is necessary to ensure consistency and energy-momentum balance. Indeed, despite the correction (second) term in the total magnetization

$$\mathbf{M}_{\rm gc}(\mathbf{X},t) \equiv -\int \left[\mu\widehat{\mathsf{b}} + \frac{p_\|}{B}\,\widehat{\mathsf{b}}\times\left(\widehat{\mathsf{b}}\times\frac{{\rm d}_{\rm gc}\mathbf{X}}{{\rm d}t}\right)\right] f_\mu\, dp_\|\, d\mu,$$

was known to appear in Ampère's Law [2, 5], its specific relation to energy and momentum balance has often been overlooked. Here, we present three different variational formulations of the guiding-center Vlasov-Maxwell system to show how the magnetization correction is naturally captured by the variational principles of guiding-center motion. The first principle is in terms of Lagrangian paths on the reduced (4D) guiding-center phase-space: in this context, the gyrophase coordinate is eliminated and the Vlasov distribution $f_\mu(\mathbf{X},p_\|)$ retains parametric dependence on the magnetic moment $\mu$. The second principle is purely Eulerian and exploits the structure of the extended (6D) phase-space, in which the particle energy is conjugate to the time coordinate [1]. Upon using Noether's theorem, this approach is used to identify the fluxes of the conserved energy and momentum. Also, the conservation of toroidal angular momentum is presented in the same framework. Lastly, the variational principle for the Lagrangian phase-space paths is reduced into a purely Eulerian form by using Euler-Poincaré reduction theory: the relabeling symmetry is used to pull back the Lagrangian paths into Eulerian coordinates, thereby generating an Eulerian variational principle on the reduced (4D) phase-space. We show how the specific Hamiltonian features of the particle flow are naturally encoded by combining the phase-space Lagrangian with Euler-Poincaré reduction, so that the conservation of the symplectic form and its Liouville measure follows from the Noether theorem corresponding to the relabeling symmetry.

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