The guiding-center Vlasov-Maxwell system: variational approaches*

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Abstract

The Lagrangian, Eulerian, and Euler-Poincaré variational principles for the guiding-center Vlasov-Maxwell equations are presented. Each variational principle presents a different approach to deriving guiding-center polarization and magnetization effects. The conservation laws of energy and momentum are also derived by Noether method, where the symmetric stress tensor is now shown explicitly.

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1. Guiding-center (GC) theory & Maxwell equations

- Over the decades, the variational formulation has been most successful in capturing nonlinear GC effects, as they arise from Hamiltonian perturbation theory.
- Following Littlejohn's phase-space Lagrangian for GC motion, the variational approach was exploited in [1] to establish a **self-consistent GC-Maxwell theory**.
- Although GC-Maxwell coupling also appears in recent studies, the explicit terms in **Ampère's law** have not been considered in terms of energy/momentum balance.
- For example, the total magnetization

$$\mathbf{M}_{\rm gc}(\mathbf{X},t) \equiv -\int\left[\mu\widehat{\mathsf{b}}+\frac{p_\|}{B}\,\widehat{\mathsf{b}}\times\left(\widehat{\mathsf{b}}\times\frac{{\rm d}_{\rm gc}\mathbf{X}}{{\rm d}t}\right)\right]F_\mu\,dp_\|\,d\mu,$$

carries a ***moving-dipole correction***, whose role has often been overlooked, e.g. in hybrid MHD models [2].

2. Three different variational principles

- In [3], we approach GC-Maxwell coupling by presenting the corresponding Vlasov kinetic theory, as it emerges from three different variational principles (VP's):
  1. **Lagrangian**: in terms of Lagrangian paths;
  2. **Eulerian**: extended Eulerian coordinates;
  3. **Euler-Poincaré**: relabeling transformations.
- The magnetic moment invariant and the ignorable gyrophase coordinate are eliminated to express the Vlasov density $F_\mu(\mathbf{X},p_\|)$ on the reduced (4D) phase-space
- Unlike previous studies, no $E\times B$ comoving frame is introduced, so polarization effects do not appear explicitly.

3. The Lagrangian variational principle

- Following the works by Low and Sugama, we introduce the phase-space VP in terms of **Lagrangian paths**:

$$z^a(t;\mathbf{z}_0,\mu)\;\equiv\;\big(\mathbf{X}(t;\mathbf{z}_0,\mu),\,p_\|(t;\mathbf{z}_0,\mu)\big),$$

parameterized by the magnetic moment invariant μ.
- In CGS units, the corresponding Lagrangian action reads

$$\mathcal{A}^{\rm L}_{\rm gc}=\int_{t_1}^{t_2}\left\{\int F_0(\mathbf{z}_0,\mu)\left[\frac{e}{c}\mathbf{A}^*\cdot\dot{\mathbf{X}}-\left(\frac{p_\|^2}{2m}+e\,\Phi^*\right)\right]d^4z_0\,d\mu+\int\frac{d^3x}{8\pi}\left(|\mathbf{E}|^2-|\mathbf{B}|^2\right)\right\}dt$$

with $\mathbf{B}=\nabla\times\mathbf{A}$ and $\mathbf{E}=-\nabla\Phi-c^{-1}\partial_t\mathbf{A}$, as well as

$$e\,\Phi^*=e\,\Phi+\mu\,B\,,\qquad e\,\mathbf{A}^*=e\,\mathbf{A}+c\,p_\|\widehat{\mathsf{b}}\,.$$

- Taking variations δz^a yields

$$\mathbf{E}^*+\frac{e}{c}\dot{\mathbf{X}}\times\mathbf{B}^*=\dot{p}_\|\,\widehat{\mathsf{b}}\,,\qquad \widehat{\mathsf{b}}\cdot\dot{\mathbf{X}}=\frac{p_\|}{m}$$

while taking variations $(\delta\mathbf{A},\delta\Phi)$ returns

$$\nabla\cdot\mathbf{E}=4\pi\int F_0\,\delta(\mathbf{x}-\mathbf{X})\,d^4z_0\,d\mu$$

$$\nabla\times\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}=4\pi\left\{\frac{e}{c}\int\dot{\mathbf{X}}\,F_0\,\delta(\mathbf{x}-\mathbf{X})\,d^4z_0\,d\mu-\nabla\times\int\left[\mu\widehat{\mathsf{b}}+\frac{p_\|}{B}\,\widehat{\mathsf{b}}\times\left(\widehat{\mathsf{b}}\times\dot{\mathbf{X}}\right)\right]F_0\,\delta(\mathbf{x}-\mathbf{X})\,d^4z_0\,d\mu\right\}.$$

Notice the GC dipole moment contribution $(m/B)\,\widehat{\mathsf{b}}\times\dot{\mathbf{X}}$.

4. Eulerian VP on the extended phase-space

- In [4], an Eulerian VP was formulated for the Maxwell-Vlasov system. This is now extended to GC-Maxwell.
- One **extends the GC phase space** by including the conjugate coordinates (t,w), where w is the particle energy. In this extended phase-space, the Eulerian VP reads

$$\mathcal{A}^{\rm E}_{\rm gc}=\int\frac{d^3x\,dt}{8\pi}\left(|\mathbf{E}|^2-|\mathbf{B}|^2\right)+\int\mathcal{F}_\mu\left[\left(\frac{1}{2m}\mathsf{p}_\|^2+e\,\Phi^*\right)-w\right]d^6Z\,d\mu\,,$$

where $\mathcal{F}_\mu$ depends on the coordinates $Z^\alpha=(\mathbf{X},\mathsf{p}_\|,t,w;\mu)$.
- The noncanonical structure of GC dynamics leads to the first two terms in the variations $\delta\mathcal{F}_\mu$, which are written as

$$\delta\mathcal{F}_\mu=\frac{e}{c}\delta\mathbf{A}^*\cdot\left(B_\|^*\left\{\mathbf{X},\,\mathcal{F}_\mu/B_\|^*\right\}_{\rm gc}\right)+\delta B_\|^*\,\mathcal{F}_\mu/B_\|^*+B_\|^*\,\left\{\delta\mathcal{S},\,\mathcal{F}_\mu/B_\|^*\right\}_{\rm gc}.$$

Here, $\delta\mathcal{S}$ is arbitrary and we use the Poisson bracket

$$\{\mathcal{F},\,\mathcal{G}\}_{\rm gc}=\frac{\mathbf{B}^*}{B_\|^*}\cdot\left(\nabla^*\mathcal{F}\,\frac{\partial\mathcal{G}}{\partial\mathsf{p}_\|}-\frac{\partial\mathcal{F}}{\partial\mathsf{p}_\|}\,\nabla^*\mathcal{G}\right)-\frac{c\,\widehat{\mathsf{b}}}{eB_\|^*}\cdot\nabla^*\mathcal{F}\times\nabla^*\mathcal{G}+\left(\frac{\partial\mathcal{F}}{\partial w}\frac{\partial\mathcal{G}}{\partial t}-\frac{\partial\mathcal{F}}{\partial t}\frac{\partial\mathcal{G}}{\partial w}\right)$$

as well as the notation $\nabla^*\mathcal{F}=\nabla\mathcal{F}-(e/c)\partial_t\mathbf{A}^*\partial_w\mathcal{F}$.
- Stationarity with respect to $\delta\mathcal{S}$ yields $B_\|^*\{\mathcal{F}_\mu/B_\|^*,\mathcal{H}\}_{\rm gc}=0$. Then, upon integrating over w, replacing the ansatz

$$\mathcal{F}_\mu\;\equiv\;F_\mu(\mathbf{X},\mathsf{p}_\|,t)\,\delta(w-H_{\rm gc}),$$

leads to the Vlasov equation in the form

$$\frac{\partial F_\mu}{\partial t}+\frac{\partial}{\partial\zeta^a}\left(F_\mu\,\frac{d_{\rm gc}\zeta^a}{dt}\right)=0\qquad\text{with } \zeta^a=(\mathbf{X},\mathsf{p}_\|)\,.$$

- Variations with respect to $(\delta\Phi,\delta\mathbf{A})$ yield Gauss' and Ampère's laws in the Eulerian form

$$\nabla\cdot\mathbf{E}=4\pi\int F_\mu\,d\mathsf{p}_\|\,d\mu$$

$$\nabla\times\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}=4\pi\left(\nabla\times\mathbf{M}_{\rm gc}+\frac{e}{c}\int\frac{{\rm d}_{\rm gc}\mathbf{X}}{{\rm d}t}\,F_\mu\,d\mathsf{p}_\|\,d\mu\right).$$

5. Euler-Poincaré VP and relabeling transformations

- The action $\mathcal{A}^{\rm L}_{\rm gc}$ can be rewritten in terms of Eulerian variables by the same process as in inviscid fluid dynamics.
- This process is known as Euler-Poincaré reduction by symmetry and it has been successfully applied to various contexts, including plasma kinetic theories [5].
- As the Lagrangian paths $z^a(t,\mathbf{z}_0;\mu)$ are smooth and invertible, they are regarded as $\boldsymbol{\mu}$**−dependent relabeling transformations** taking the label $\mathbf{z}_0$ to its current position.
- Upon defining the Lagrange-to-Euler map (push-forward)

$$F_\mu(\mathbf{X},\mathsf{p}_\|)=\int F_0(\mathbf{z}_0;\mu)\,\delta(\zeta^a-z^a(t,\mathbf{z}_0;\mu))\,d^4z_0$$

the Lagrangian action $\mathcal{A}^{\rm L}_{\rm gc}$ is rewritten in the Eulerian form

$$\mathcal{A}^{\rm EP}_{\rm gc}=\int_{t_1}^{t_2}\left[\int F_\mu(\boldsymbol{\zeta})\left(\frac{e}{c}\mathbf{A}^*\cdot\mathbf{U}(\boldsymbol{\zeta})-\frac{\mathsf{p}_\|^2}{2m}-e\,\Phi^*\right)d^4\zeta\,d\mu+\int\left(|\mathbf{E}(\mathbf{x})|^2-|\mathbf{B}(\mathbf{x})|^2\right)\frac{d^3x}{8\pi}\right]dt,$$

- Here, one defines the vector field

$$\Xi(\boldsymbol{\zeta},t)=\Big(\mathbf{U}(\boldsymbol{\zeta},t),\,\phi_\|(\boldsymbol{\zeta},t)\Big)\,,$$

such that $\dot{z}^a=\Xi^a(z,t)$. Then, one finds

$$\delta\Xi=\frac{\partial\Upsilon}{\partial t}+\Upsilon^a\frac{\partial\Xi}{\partial\zeta^a}-\Xi^a\frac{\partial\Upsilon}{\partial\zeta^a},\qquad \delta F_\mu=-\frac{\partial}{\partial\zeta^a}\left(\Upsilon^a\,F_\mu\right),$$

where Υ is arbitrary and vanishes at the endpoints.

- Taking the variations $(\delta\Xi,\delta F_\mu)$ in $\mathcal{A}^{\rm EP}_{\rm gc}$ yields

$$\mathbf{U}=\frac{{\rm d}_{\rm gc}\mathbf{X}}{{\rm d}t}=\frac{1}{B_\|^*}\left(\frac{\mathsf{p}_\|}{m}\mathbf{B}^*+c\mathbf{E}^*\times\widehat{\mathsf{b}}\right)$$

$$\phi_\|=\frac{{\rm d}_{\rm gc}p_\|}{{\rm d}t}=\frac{e}{B_\|^*}\mathbf{E}^*\cdot\mathbf{B}^*,$$

along with the Vlasov equation

$$\frac{\partial F_\mu}{\partial t}+\frac{\partial}{\partial\mathsf{X}^i}(U^i\,F_\mu)+\frac{\partial}{\partial\mathsf{p}_\|}(\phi_\|F_\mu)=0\,.$$

- Again, variations with respect to $(\delta\Phi,\delta\mathbf{A})$ yield Gauss' and Ampère's laws in the Eulerian form.

6. Noether conservation laws

- The symmetry properties of Hamilton's VP lead to conservation laws, according to Noether's theorem.
- The time-reversal symmetry of the action $\mathcal{A}^{\rm E}_{\rm gc}$ yields **energy conservation** $\partial_t\mathcal{E}+\nabla\cdot\mathbf{S}_{\rm gc}=0$ with the definitions

$$\mathcal{E}_{\rm gc}\equiv\int F_\mu\left(\frac{1}{2m}\mathsf{p}_\|^2+\mu B\right)d\mathsf{p}_\|d\mu+\frac{1}{8\pi}\left(|\mathbf{E}|^2+|\mathbf{B}|^2\right),$$

$$\mathbf{S}_{\rm gc}\equiv\int F_\mu\left(\frac{1}{2m}\mathsf{p}_\|^2+\mu B\right)\frac{d_{\rm gc}\mathbf{X}}{dt}\,d\mathsf{p}_\|d\mu+\frac{c}{4\pi}\,\mathbf{E}\times\mathbf{H}.$$

- Analogously, the translational symmetry of the action $\mathcal{A}^{\rm E}_{\rm gc}$ yields the **momentum balance** $\partial_t\mathbf{P}_{\rm gc}+\nabla\cdot\mathsf{T}_{\rm gc}=0$, where

$$\mathbf{P}_{\rm gc}=\int\mathsf{p}_\|\,\widehat{\mathsf{b}}\,F_\mu\,d\mathsf{p}_\|d\mu+\frac{\mathbf{E}\times\mathbf{B}}{4\pi\,c}$$

$$\mathsf{T}_{\rm gc}=\frac{1}{8\pi}\left(|\mathbf{E}|^2+|\mathbf{B}|^2\right)-\frac{1}{4\pi}\left(\mathbf{E}\mathbf{E}+\mathbf{B}\mathbf{B}\right)+\int\left\{\mathsf{p}_\|\left[\widehat{\mathsf{b}}\otimes_s\left(\frac{d_{\rm gc}\mathbf{X}}{dt}\right)_\perp\right]+\left[\frac{1}{m}\mathsf{p}_\|^2\,\widehat{\mathsf{b}}\,\widehat{\mathsf{b}}+\mu B\left(\mathbf{I}-\widehat{\mathsf{b}}\widehat{\mathsf{b}}\right)\right]\right\}F_\mu\,d\mathsf{p}_\|d\mu.$$

Here, we have introduced the notation $\mathbf{v}\otimes_s\mathbf{w}=\mathbf{v}\mathbf{w}+\mathbf{w}\mathbf{v}$.
- Also, symmetry under rotations around the vertical axis yields conservation of the **toroidal angular momentum**:

$$\frac{\partial}{\partial t}\left(\int F_\mu\,P_\varphi\,d\mathsf{p}_\|d\mu\right)+\nabla\cdot\left(\int\frac{{\rm d}_{\rm gc}\mathbf{X}}{{\rm d}t}F_\mu\,P_\varphi\,d\mathsf{p}_\|d\mu\right)=0$$

- Finally, the Lagrangian action $\mathcal{A}^{\rm L}_{\rm gc}$ is invariant under relabeling transformations that preserve the reference density $F_0(\mathbf{z}_0;\mu)$. Then the **symplectic form** is conserved

$$\frac{d}{dt}\iint_{\sigma(t;\mu)}{\rm d}\mathbf{X}\wedge{\rm d}\mathbf{P}=0\,,\qquad\text{with}\qquad \mathbf{P}=\frac{e}{c}\mathbf{A}(\mathbf{X},t)+\mathsf{p}_\|\widehat{\mathsf{b}}(\mathbf{X},t)$$

for any surface $\sigma\subset\mathbb{R}^4$ moving with the flow $z^a(t,\mathbf{z}_0;\mu)$.
- As a consequence, the **Liouville density** is conserved:

$$\frac{\rm d}{{\rm d}t}\Big(mB_\|^*(\mathbf{X})\,d^3\mathsf{X}\,d\mathsf{p}_\|\Big)=0$$

7. Summary

- Three different variational principles were presented for the GC Vlasov-Maxwell system.
- The symmetry properties of the actions were used to produce explicit conservation laws for energy & momentum
- The moving dipole contribution in the magnetization is *determinant* to ensure energy and momentum balance.
- The energy flux was presented along with the stress tensor, which was found to be symmetric (as often assumed).

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