Analytical theory and numerical investigation of the shear Alfvén continuum in the presence of an island

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Overview

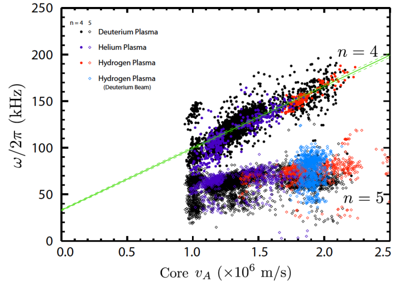
1. Experimental motivation from unexplained Alfvén eigenmodes (AEs) in MST

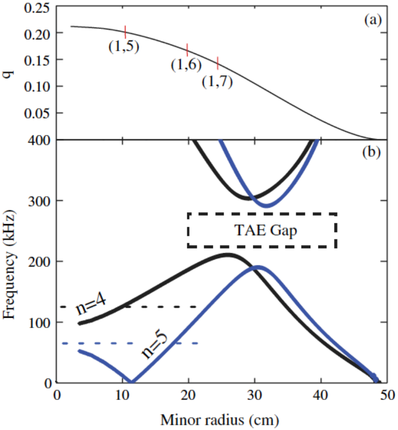
2. Derive Alfvén continuum in the presence of an island
   - Introduce an island coordinate system
   - Calculate the spectrum using WKB theory
   - Demonstrate the existence of a continuum accumulation point at the separatrix

3. Compare theory to MST observations
   - Analytical continuum comparison to MST bursts
   - Numerical simulation of continuum using STELLGAP
   - Identification of Island-induced Alfvén Eigenmode (IAE) using AE3D code

4. Future work using SIESTA and summary

Motivation: Unexplained Alfvénic activity has been observed during NBI in the MST reversed field pinch

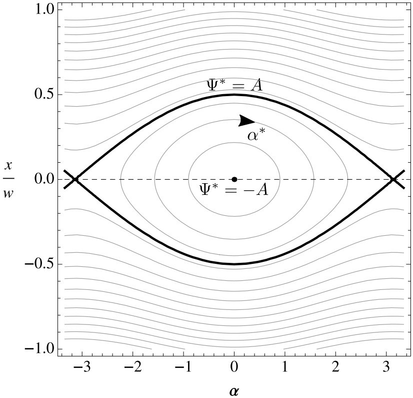




[1]Koliner, 2012 PRL.

*Previous numerical studies neglected the presence of a large $m = 1$, $n = 5$ island in the core of MST.

Island coordinate system labels flux surfaces inside and outside the separatrix



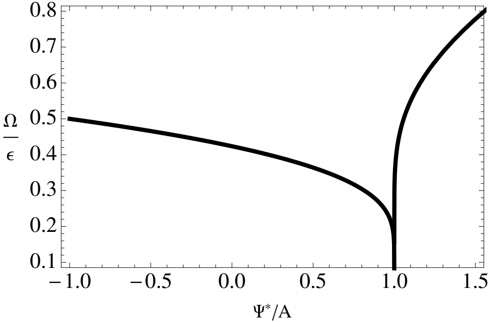
$$\begin{aligned}
\Psi^* &= \int d\psi(q - q_0) - A\cos(n_0\alpha) \\
&\approx q_0' \frac{x^2}{2} - A\cos(n_0\alpha) \\
\chi &= \theta \\
\alpha &= \zeta - q_0\theta
\end{aligned}$$

[2]Hegna, Callen, 1992 Phys. Fluids B.

Here $x = \psi - \psi_0$ is the distance from the rational surface and $q_0' = dq/d\psi|_{\psi=\psi_0}$. The magnetic island width is given by $w = 4\sqrt{|A/q_0'|}$.

Island rotational transform

Additionally, $\Phi^*(\Psi^*)$, α^*, and an island rotational transform $\Omega = d\Psi^*/d\Phi^*$ can be defined.



Inside the separatrix:

$$\Omega(\Psi^*) = \frac{\pi\epsilon}{4K(\kappa^2)}$$

$$\kappa^2 = \frac{\Psi^* + A}{2A}$$

Outside the separatrix:

$$\Omega = \pm\frac{\pi\epsilon}{2kK(k^2)}$$

$$k^2 = \frac{2A}{|\Psi^*| + A}$$

$\epsilon = q_0' w/2 \ll 1$
is a small parameter

Island coordinates give straight field-line form for **B**

In the island coordinate system, the total magnetic field can be written in a straight field form as

$$\mathbf{B} = \mathbf{B_0} + \mathbf{B_1} = \nabla\alpha^* \times \nabla\Phi^* + \Omega\nabla\Phi^* \times \nabla\chi$$

This allows us to write the derivative along the magnetic field line in a much simpler form:

$$\mathbf{B} \cdot \nabla\lambda = \frac{1}{\sqrt{g}} \left[\frac{\partial\lambda}{\partial\chi} + \Omega \frac{\partial\lambda}{\partial\alpha^*} \right]$$

Linearized ideal MHD

Equilibrium satisfies force balance, Ampére's Law, and the divergence constraint:

$$\begin{aligned} \mathbf{J} \times \mathbf{B} &= \nabla p \\ \nabla \times \mathbf{B} &= \mathbf{J} \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned}$$

The linearized ideal MHD equations are the momentum equation, the equation of state, and the combined Faraday's law/Ohm's law (natural units for plasmas, $\mu_0, \epsilon_0 = 1$, are used):

$$\begin{aligned} -\rho\omega^2 \boldsymbol{\xi} &= (\nabla \times \mathbf{B}) \times \boldsymbol{\delta}\mathbf{B} + (\nabla \times \boldsymbol{\delta}\mathbf{B}) \times \mathbf{B} - \nabla \delta p \\ 0 &= \delta p + \boldsymbol{\xi} \cdot \nabla p + \gamma p \nabla \cdot \boldsymbol{\xi} \\ \boldsymbol{\delta}\mathbf{B} &= \nabla \times (\boldsymbol{\xi} \times \mathbf{B}) \end{aligned}$$

Shear Alfvén continuum equation in the presence of an island

Following Cheng and Chance's treatment[3], using the variables $\nabla \cdot \boldsymbol{\xi}$, $\xi_{\Phi^*} = \boldsymbol{\xi} \cdot \nabla \Phi^*$, $\delta P = \delta p + \boldsymbol{\delta} \mathbf{B} \cdot \mathbf{B}$, and $\xi_s = \boldsymbol{\xi} \cdot (\mathbf{B} \times \nabla \Phi^*)/|\nabla \Phi^*|^2$, the Alfvén continuum equation in a cylinder with an island is

$$\omega^2 \rho \frac{|\nabla \Phi^*|^2}{B^2} \xi_s + \mathbf{B} \cdot \nabla \left(\frac{|\nabla \Phi^*|^2}{B^2} \mathbf{B} \cdot \nabla \xi_s \right) = 0$$

Using the parallel gradient operator in island coordinates results in the following second-order ODE:

$$\frac{d}{d\alpha^*} \left(|\nabla \Phi^*|^2 \frac{d}{d\alpha^*} \xi_s \right) + \omega^2 \rho \frac{\sqrt{g}^2}{\Omega^2} |\nabla \Phi^*|^2 \xi_s = 0$$

Here, no variation in the χ direction is assumed.

WKB analysis of continuum equation

The continuum equation can be rewritten using $|\nabla\Phi^*|^2 = (q_0' x/\Omega)^2 |\nabla\psi|^2$ and a normalized frequency $\hat{\omega}^2 = \omega^2/\omega_A^2$, where the Alfvén frequency is $\omega_A = 1/\sqrt{\rho}\sqrt{g}$:

$$\frac{\Omega^2}{\hat{\omega}^2}\hat{x}^2 \frac{d}{d\alpha^*}\left[\hat{x}^2 \frac{d}{d\alpha^*}\xi_s\right] + \hat{x}^4 \xi_s = 0$$

Here $\hat{x} = \sqrt{\kappa^2 - \sin^2 n_0\alpha/2}$ and $x = (w/2)\hat{x}$.
The boundary conditions are periodic in α^*:

$$\begin{aligned} \xi_s(\alpha^*) &= \xi_s(\alpha^* + 2\pi) \\ \left.\frac{d\xi_s}{d\alpha^*}\right|_{\alpha^*} &= \left.\frac{d\xi_s}{d\alpha^*}\right|_{\alpha^*+2\pi} \end{aligned}$$

Comparison to the canonical Schrödinger equation, $\delta^2 d^2Y/dt^2 - Q(t)Y = 0$, identifies $\delta = \Omega/\hat{\omega} \sim \epsilon$ as the small parameter. A second-order WKB expansion in δ is used to solve for the continuum.

Eigenmodes inside and outside the island

Inside the island, the periodic boundary conditions require eigenmodes that are odd for j_{in} odd and even for j_{in} even:

$$
\begin{aligned}
\xi_O^j &= \frac{1}{\sqrt{\pi}\hat{x}(\alpha^*)} \sin\left[n_0(j_{in}+1)\alpha^*\right],\ j_{in} = 1, 3, 5, ... \\
\xi_E^j &= \frac{1}{\sqrt{\pi}\hat{x}(\alpha^*)} \cos\left[n_0(j_{in}+1)\alpha^*\right],\ j_{in} = 2, 4, 6, ...
\end{aligned}
$$

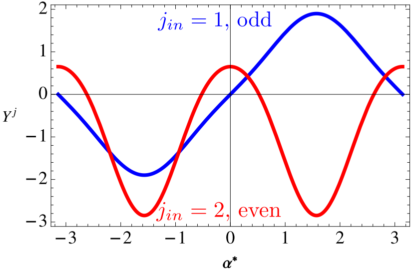
Outside the island, modes of both parities are allowed for all quantum numbers, giving the following eigenmodes:

$$
\begin{aligned}
\xi_O^j &= \frac{1}{\sqrt{\pi}\hat{x}(\alpha^*)} \sin\left[j_{out}\alpha^*\right] \\
\xi_E^j &= \frac{1}{\sqrt{\pi}\hat{x}(\alpha^*)} \cos\left[j_{out}\alpha^*\right]
\end{aligned}
$$

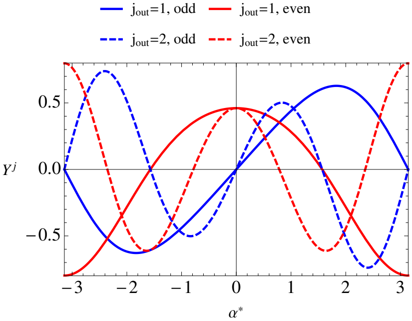
Here $j_{out} = 1, 2, 3, ...$

Continuum mode structure

Continuum mode structure for $\Psi^* = A/2$ surface inside the island



Continuum mode structure for $\Psi^* = 2A$ surface outside the island



Shear Alfvén dispersion relation in the presence of an island

The eigenfrequencies $\hat{\omega}$ corresponding to the continuum eigenmodes are given by

$$\hat{\omega}^2 = \left[\frac{j\Omega}{2} + \sqrt{\left(\frac{j\Omega}{2}\right)^2 + \frac{q_0'}{2}\left(\Psi^* - \frac{1}{2}\Omega\Phi^*\right)}\right]^2$$

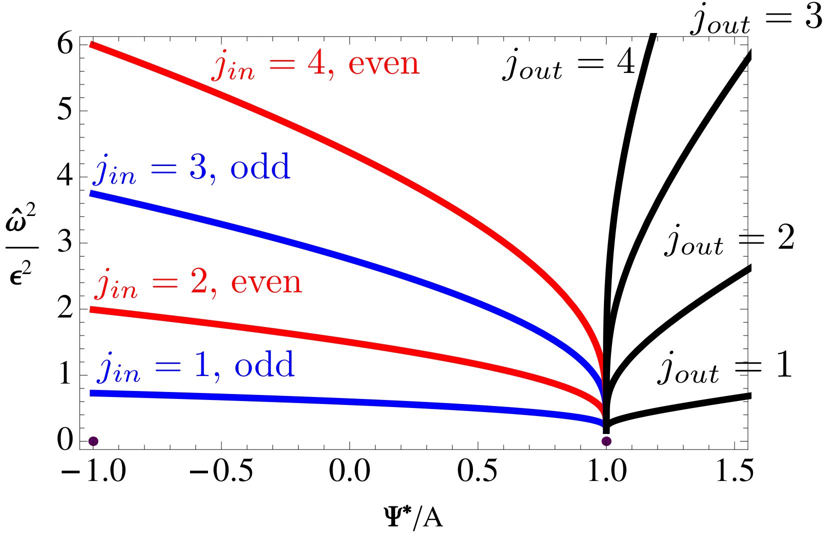
Inside the separatrix:

$$\begin{aligned} j &= n_0(j_{in} + 1) \\ j_{in} &= 1, 2, 3, \ldots \end{aligned}$$

Outside the separatrix:

$$\begin{aligned} j &= j_{out} \\ j_{out} &= 1, 2, 3, \ldots \end{aligned}$$

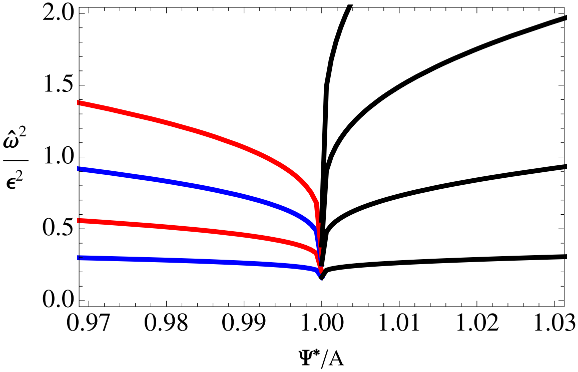
Shear Alfvén continuum frequencies



All branches converge to an upshifted continuum accumulation point (CAP) at the separatrix

$$\left(\frac{\hat{\omega}_{sep}}{\epsilon}\right)^2 = \lim_{\Psi^* \to A} \left(\frac{\hat{\omega}}{\epsilon}\right)^2 = \frac{1}{8}$$

$$\omega_{sep}^2 = (q_0' w)^2 \omega_A^2 / 32$$

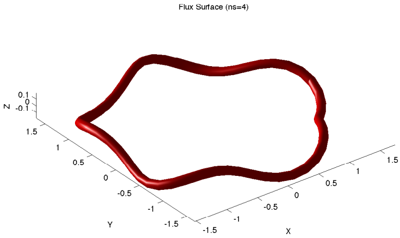


This analytic upshift agrees with previous numerical results by Biancalani et al[4]

Non-reversed plasmas in MST contain a sizable $n = 5$ island that can be modeled in VMEC as a SHAx state

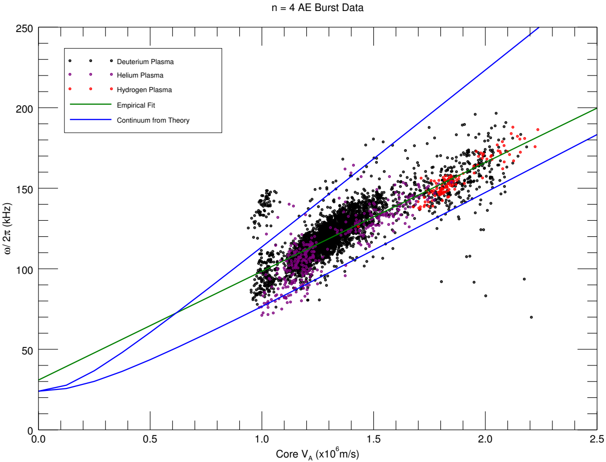
MST parameters:

$$
\begin{aligned}
R_0 &= 1.5m \\
a &= .5m \\
I_p &= 200 - 500kA \\
n_i &\cong .7 \times 10^{19}/m^3 \\
q_0' &\cong .004/cm \\
\frac{w}{2} &\cong 7cm \\
\epsilon &= \frac{q_0' w}{2} \cong .028
\end{aligned}
$$



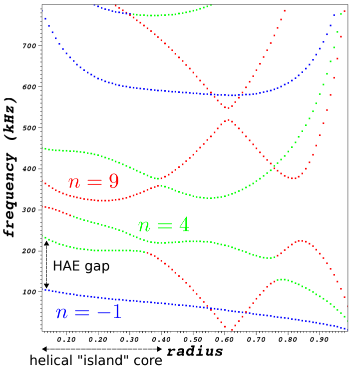
Observed $n = 4$ AE bursts during NBI in MST lie in the continuum gap from analytical theory

The AE burst frequencies observed in MST are enveloped by the analytic continuum frequencies in the core of the island

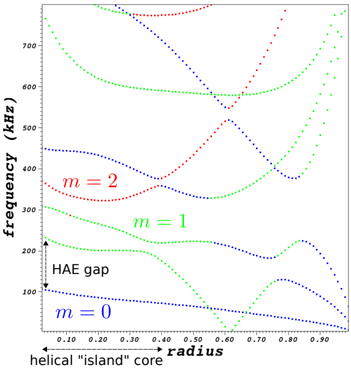


STELLGAP continuum calculations predict $n = 4$, $n = -1$-coupled gap in island core

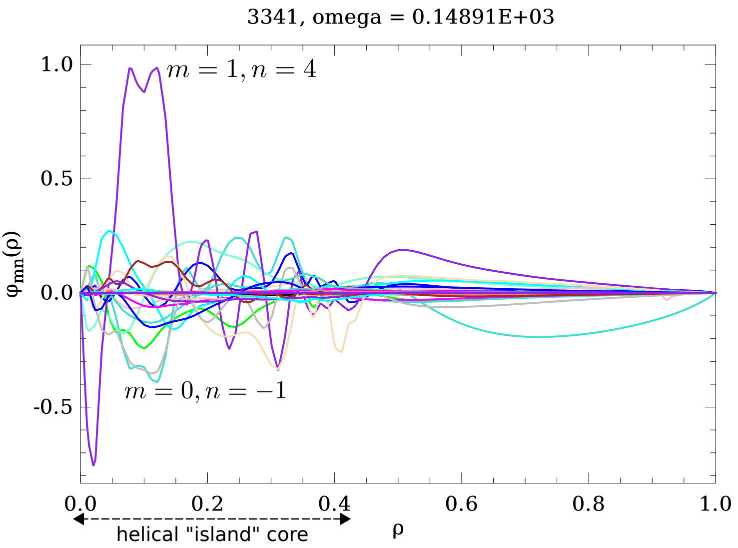
Continuum frequencies plotted with dominant n mode number



Continuum frequencies plotted with dominant m mode number



$n = 4$ AE mode discovered with AE3D code at 149 kHz, in close agreement to 151 kHz experimentally



This appears to be the first identification of an Island-induced Alfvén Eigenmode (IAE)

The Island-induced Alfvén Eigenmode has similar mode coupling to a Helicity-induced Alfvén Eigenmode (HAE), but is localized to the core of an island. The dominant components of the computed AE are $m = 1$, $n = 4$ and $m = 0$, $n = -1$.

HAE coupling:

$$\begin{aligned} \delta_n &= \pm n_0 \\ \delta_m &\geq 1 \end{aligned}$$

Computed AE coupling:

$$\begin{aligned} \delta_n &= 5 \\ \delta_m &= 1 \end{aligned}$$

This AE3D simulation was done with a VMEC SHAx equilibrium. To truly confirm this AE as an IAE, an equilibrium containing an island (and two axes) must be used.

SIESTA can now be used to study the Alfvén spectrum in equilibria containing islands

SIESTA is an iterative equilibrium solver capable of resolving magnetic islands.[5]

Recently, inertia was added to SIESTA. At each iteration, SIESTA solves the following system for a new plasma displacement $\boldsymbol{\xi}$:

$$-\omega^2 \mathbf{T}\boldsymbol{\xi} = \mathbf{F_0} + \mathbf{H}\boldsymbol{\xi}$$

Here $\mathbf{H}$ is the Hessian matrix, $\mathbf{F_0}$ is the force residual of the current iteration, and $\mathbf{T} = \rho \mathbf{g}_{ij}$ is the inertia matrix, where ρ is the mass density and $\mathbf{g}_{ij}$ are the lower metric elements. Once SIESTA converges to an equilibrium $\mathbf{F_0} \to 0$, the system becomes

$$-\omega^2 \mathbf{T}\boldsymbol{\xi} = \mathbf{H}\boldsymbol{\xi}$$

This equation can be solved to obtain the MHD (and Alfvén) eigenmodes and frequencies.

Summary

- The Alfvén continuum in the presence of an island was studied using WKB theory
- An Alfvén continuum accumulation point upshift at the separatrix has been predicted analytically for the first time
- The continuuum branches from analytic theory provide an envelope for MST AE burst frequencies, suggesting the mode is an Island-induced Alfvén Eigenmode (IAE)
- STELLGAP/AE3D calculations confirm the theory, and a mode consistent with an Island-induced Alfvén Eigenmode has been found
- Future work will involve looking for AEs in equilibria with islands obtained with the SIESTA code

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