Applications of asymptotic-preserving (AP) methods to plasma dynamics simulations at realistic dimensionless parameters

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Abstract

Over the past decade the numerical treatment of singularly perturbed problems has advanced significantly with the advent of asymptotic-preserving (AP) techniques [1]. For problems characterized by a small parameter ϵ, AP schemes are designed to work accurately in both the limit when the parameter is of order unity and when it approaches zero. AP schemes rely on a suitably constructed implicit part to assure the correct asymptotic behavior, thereby overcoming the limitations of a generic fully implicit or semi-implicit scheme. In this work, we present two applications of AP methods to the context of plasma physics.

The first application concerns the strongly anisotropic transport of a scalar, such as temperature, in magnetic island geometry. Here the small parameter ϵ is the ratio of the perpendicular to the parallel transport coefficient. A suitable AP method is constructed which works on a uniform grid, with both open and closed field lines, without requiring alignment of the grid to the magnetic field. The strength of the method is demonstrated for values of the small parameter as low as $\epsilon = 10^{-20}$ in the case of both static and rotating magnetic islands [2].

As a second application, we have considered reduced resistive MHD system (RMHD) in two dimensions. Numerical simulations of RMHD are notoriously challenging because of the disparate time-scales, encompassing the Alfvén wave period and the resistive diffusion time, and because of the formation of thin internal layers, especially in the nonlinear phase. Upon suitable rescaling of the original equations, the small parameter ϵ turns out to be the inverse of the square of the Lundquist number S, $\epsilon = S^{-2}$. The new scheme is specifically designed to study the long time scale dynamics with large time steps on the resistive time scale. The tearing mode evolution and the formation of a magnetic island are considered as a test case. One finds that the scheme is able to reproduce efficiently the three regimes of the island dynamics, linear, Rutherford growth and saturation, with good agreement with known analytical results [3] for this problem. The scheme is shown to work well at $\epsilon = 10^{-16}$ (Lundquist number $S = 10^8$), with the quality of the simulation depending essentially on the resolution necessary to treat the small visco-resistive layer occurring around the separatrix [4].

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[2] J. Narski, M. Ottaviani , Computer Physics Communications 185, 3189 (2014).
[3] D. F. Escande and M. Ottaviani, Physics Letters A, 323, 278 (2004).
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Asymptotic Preserving (AP) methods

- Call P_ϵ a problem depending on a small parameter ϵ. Example: resistive MHD, with ϵ being the resistivity.
- Call P_0 the limiting problem as $\epsilon \to 0$. Think of ideal MHD.
- If P_0 has a different mathematical nature than P_ϵ, problems usually arise. Think again of ideal MHD.
- Call P_ϵ^h a discretised version of P_ϵ, h being the set of discretisation parameters (time step, grid spacing,...).
- Consider P_0^h, the limiting version of the discretised problem. If its solutions are consistent solutions of P_0 the scheme is called AP.
- If in addition the accuracy in the discretisation parameter h is maintained the scheme is asymptotically accurate (AA).

AP is about taking the limit in a controlled way

AP methods: many ingredients

- Judicious choice of the part of the problem to be treated implicitly
- Suitable implicit and accurate time advancing integrators, such as DIRK (diagonally implicit Runge-Kutta)
- Reformulation of the problem using auxiliary variables
- Micro-macro decomposition of the original variables in an ϵ-dependent and in an ϵ-independent part.
- Addition of suitably chosen stabilizing terms: key to limit the condition number

No black box solutions!

Growth and saturation of a magnetic island

Reduced MHD equations in 2D, *rescaled*

$$\partial_t \Delta\phi + [\phi, \Delta\phi] = \frac{1}{\epsilon}[\psi, \Delta\psi] + \Delta^2\phi$$
$$\partial_t \psi + [\phi, \psi] = \Delta\psi - \Delta\psi_e$$

Here $\epsilon = S^{-2}$ is the inverse squared Lundquist number, $\epsilon \sim 10^{-16}$, and time is normalised to the resistive time scale.

The three regimes of island evolution

a) Linear regime, exponential growth, island width $w \ll \delta$ (linear layer width)

- $\delta \sim \epsilon^{1/6} k^{-1/3}$
- $\gamma \sim 1/\delta \sim \Delta' \epsilon^{-1/6} k^{1/3}$

b) Rutherford growth, $\delta \ll w \ll w_s$ (saturation size)

- $w(t) = 1.22\Delta' t$

c) Saturation

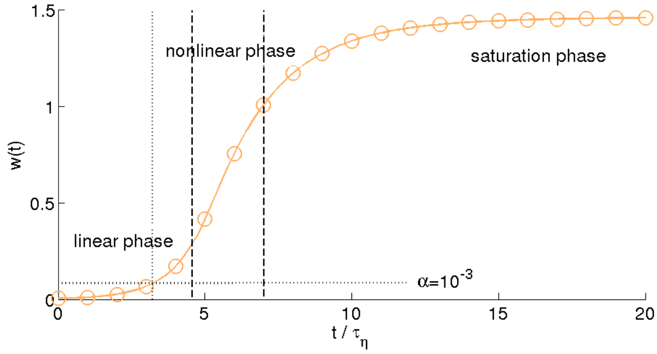
- $w_s = 2.44\Delta'$, with a coefficient dependent on the model equilibrium [3]

Generalized Rutherford's equation

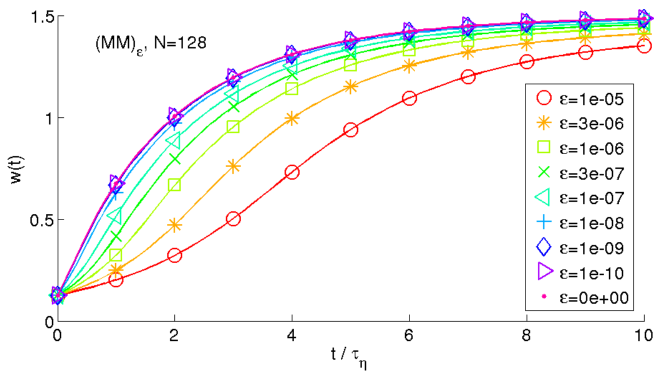
$$\frac{dw}{dt} = 1.22\,(\Delta' - \alpha w)$$

with $\alpha \approx 0.41 J_0''/J_0$

Time evolution, fixed $\epsilon = 10^{-6}$



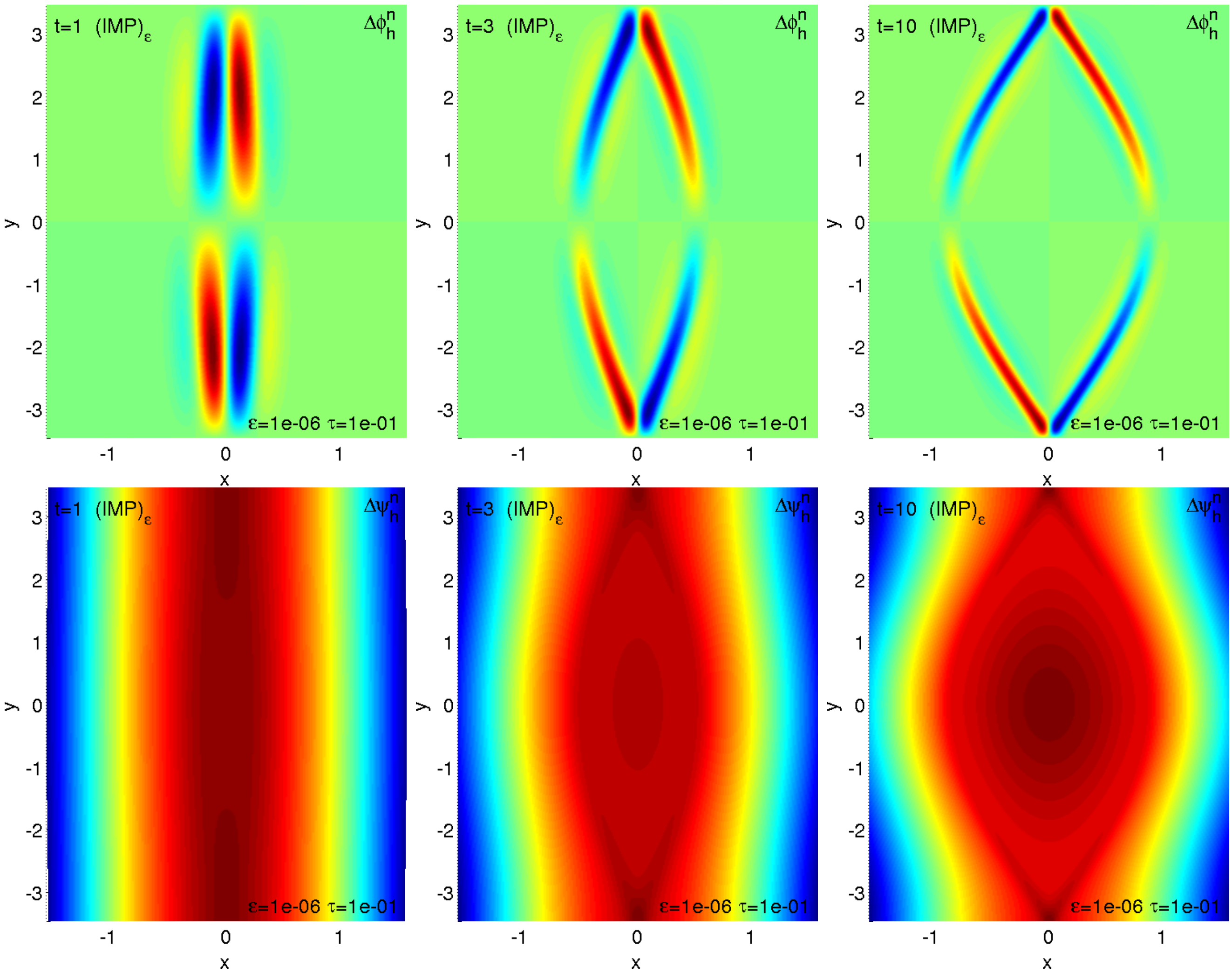
Time evolution, convergence as $\epsilon \to 0$

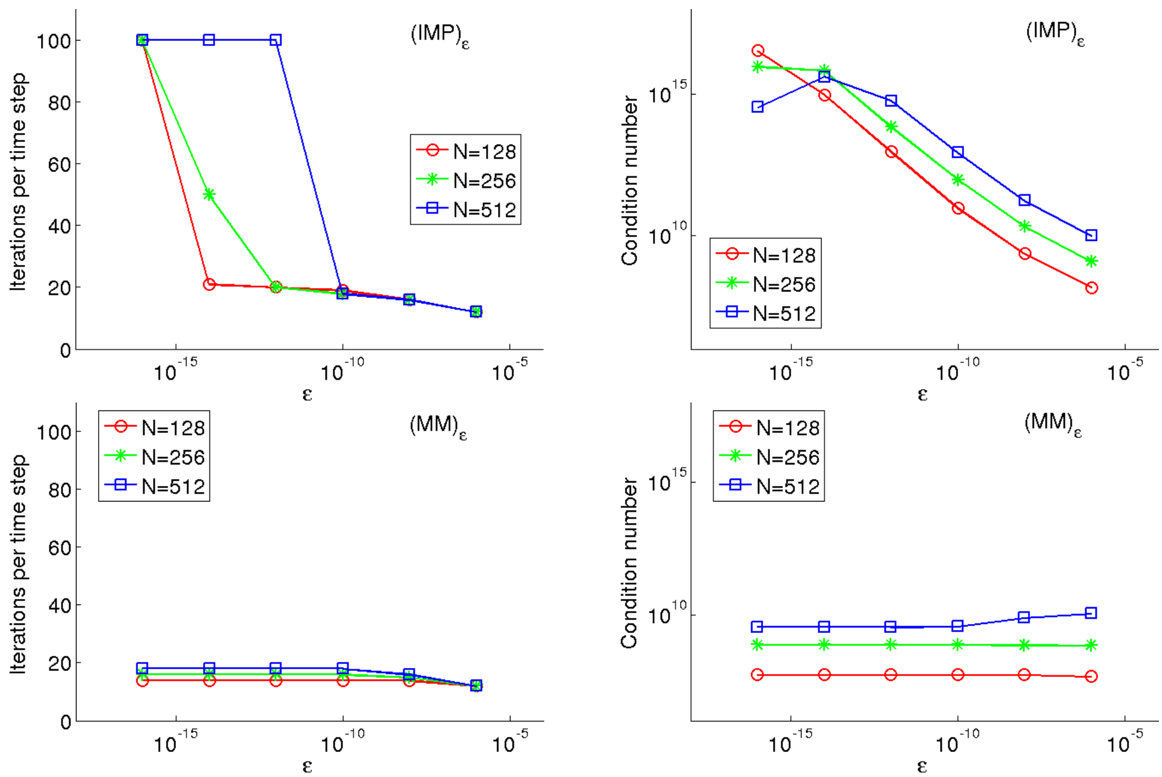


Theory vs simulation

	Δ'	$\dot{w}$ (Rutherford)	w_s
numerical result	0.66	$0.84\Delta'$	$2.24\Delta'$
analytic estimate	0.65	$1.22\Delta'$	$2.44\Delta'$

Magnetic island evolution: vorticity and current density





Number of fixed-point iterations per time step, and condition number of the system matrix as functions of ϵ, generic implicit scheme(IMP) and AP scheme (MM)

Transport in a magnetic island

Anisotropic diffusion equation

$$\partial_t T - \frac{1}{\epsilon}\nabla_\parallel \cdot (A_\parallel \nabla_\parallel T) - \nabla_\perp \cdot (A_\perp \nabla_\perp T) = 0$$

where $\epsilon \sim 10^{-10}$ is the ratio of the perpendicular to the parallel conductivity.

The magnetic field is of the form

$$\mathbf{B} = \nabla \times (\psi \hat{\mathbf{z}}) + \hat{\mathbf{z}}$$

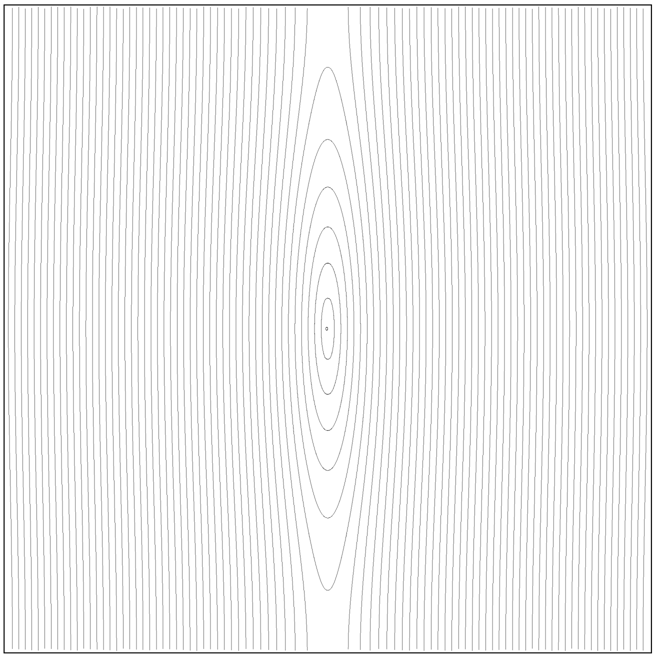
with ψ given by

$$\psi(x, y) = -\tfrac{1}{2}x^2 + \hat{b}_x \cos(k_y y - \omega t),$$

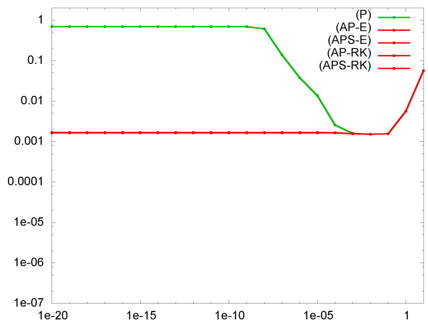
so that

$$\nabla_\parallel = x\partial_y + k_y \hat{b}_x \sin(k_y y - \omega t)\partial_x$$

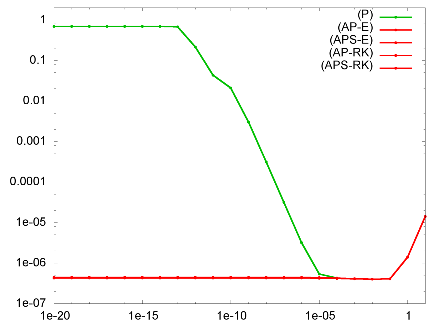
Magnetic island



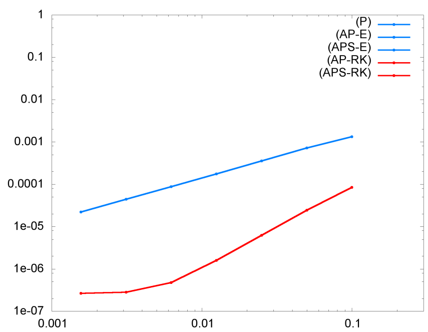
Error Analysis



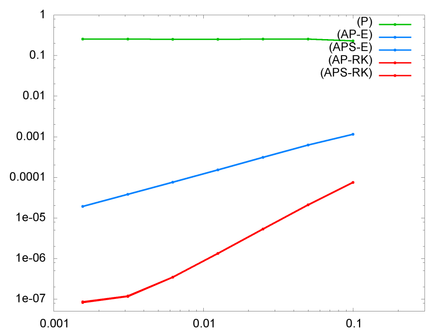
$h = 0.1$



$h = 0.00625$



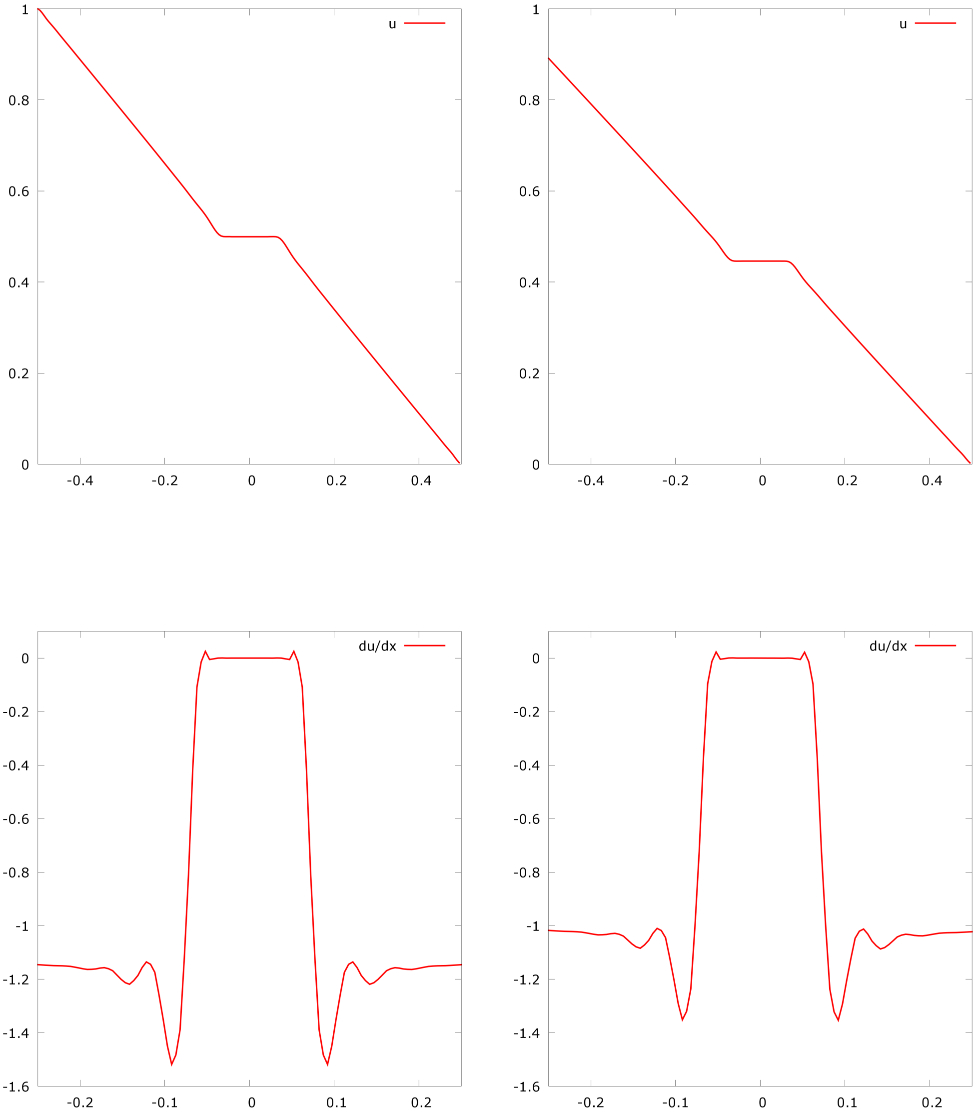
$\epsilon = 1$



$\epsilon = 10^{-20}$

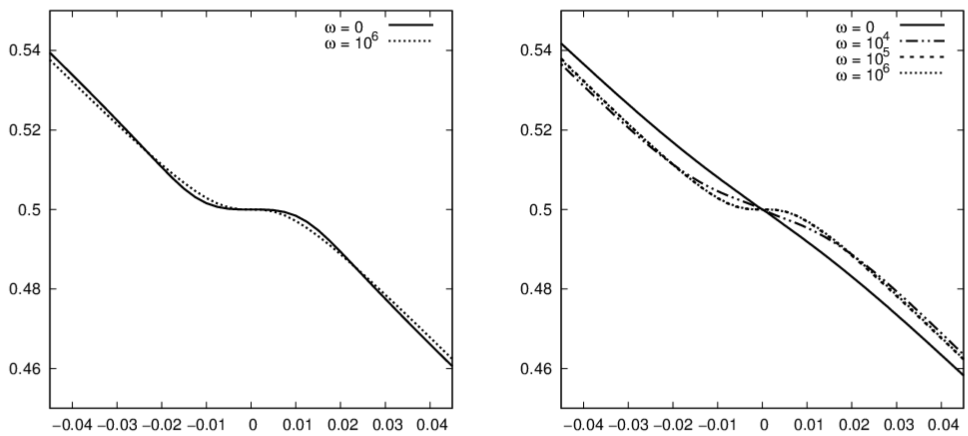
Relative L^2-errors between the exact solution and the computed solution, as a function of ϵ (above) and as a function of the timestep (below)

- (P) Standard scheme
- (E_{AP}) Euler-AP method
- (E_{APS}) Euler-APS method (stabilized)
- (RK_{AP}) DIRK-AP scheme
- (RK_{APS}) DIRK-APS scheme (stabilized)



Temperature profiles (above) and gradients (below), static island

Left: Fixed temperature difference, Dirichlet boundary conditions
Right: Fixed energy injection. Neumann boundary conditions



Temperature profiles, rotating island, various frequencies

Left: across the O-point
Right: across the X-point

The profile is progressively more and more averaged by the island rotation when the frequency increases