Accounting of Magnetic, Cross, and Kinetic Helicities in Nonlinear Two-Fluid Relaxation Simulations

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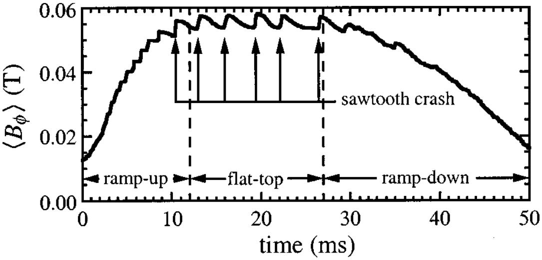
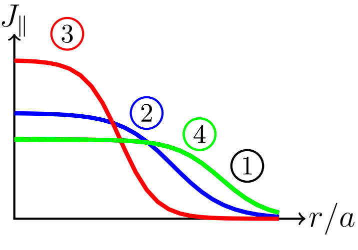
Ohmic current drive provides a source of free energy and the RFP behaves as a driven-damped system.

- Discrete flux-conversion events are observed during a discharge
- Plasma activity sustains toroidal flux against resistive decay



Den Hartog et. al. PoP **6** No. 5 (1999)



1. Current profile is initially stable
2. Ohmic current preferentially driven in the core where **B** is most aligned with **E**
3. Large gradients of $J_\parallel$ develop, destabilizing core-resonant magnetic modes
4. Core modes couple nonlinearly to edge modes and flatten $J_\parallel$

- The **sawtooth cycle** occurs multiple times in a typical RFP discharge

A variational theory based on selective decay of ideal invariants is used to predict the relaxed state.

- Taylor[1] recognized that the **magnetic helicity** ($\mathcal{K}$), a topological measure of the linkedness of magnetic field, is more robustly conserved than the magnetic energy in a resistive plasma

$$\mathcal{K} \equiv \int \mathbf{A} \cdot \mathbf{B} d^3x \qquad \frac{\partial \mathcal{K}}{\partial t} = \int \mathbf{E} \cdot \mathbf{B} d^3x \sim \int \frac{\eta}{\mu_0} \left[\sum_k k B_k^2 \right] d^3x$$

$$W_B \equiv \int \frac{\mathbf{B} \cdot \mathbf{B}}{2\mu_0} d^3x \qquad \frac{\partial W_B}{\partial t} = \int \frac{\mathbf{E} \cdot \mathbf{J}}{\mu_0} d^3x \sim \int \frac{\eta}{\mu_0} \left[\sum_k k^2 B_k^2 \right] d^3x$$

- Variational theory conserves magnetic helicity while minimizing energy

$$0 = \delta \left[W_B - \frac{\lambda}{2\mu_0} \mathcal{K} \right] = \int \frac{\delta \mathbf{A}}{\mu_0} \cdot \left[\boldsymbol{\nabla} \times \mathbf{B} - \lambda \mathbf{B} \right] d^3x \quad \rightarrow \quad \boldsymbol{\nabla} \times \mathbf{B} = \lambda \mathbf{B}$$

- Relaxed state is force-free ($\mathbf{J} \times \mathbf{B} = \mathbf{0}$) with λ a **global constant**
- The axisymmetric solution yields the Bessel function model (BFM)

$$B_z = B_0 J_0 (\lambda r) \qquad B_\theta = B_0 J_1 (\lambda r)$$

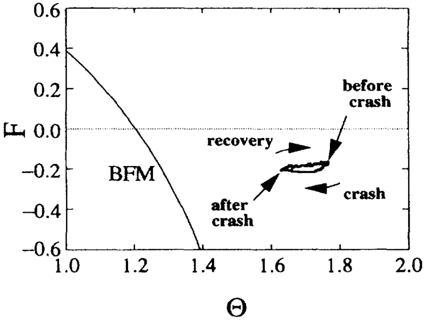
[1] Taylor, J. B. 1974. Relaxation of Toroidal Plasma and Generation of Reverse Magnetic Fields. *Physical Review Letters* **33**(19) 1139–1141

The crash phase of the sawtooth cycle brings the plasma closer to the Taylor state, but it never achieves it.

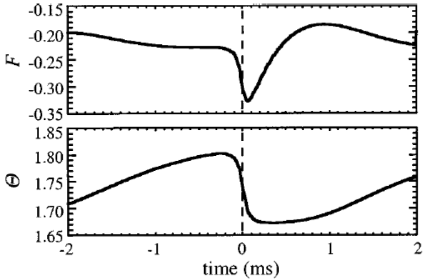
- Non-dimensionalized measures of field reversal and current drive

$$F \equiv \frac{\langle B_z \rangle|_{r=a}}{\langle B_z \rangle|_{\mathrm{vol}}}$$

$$\Theta \equiv \frac{\langle B_\theta \rangle|_{r=a}}{\langle B_z \rangle|_{\mathrm{vol}}} = \frac{\lambda a}{2}$$



Ji et. al. PRL **74** No. 15 (1995)

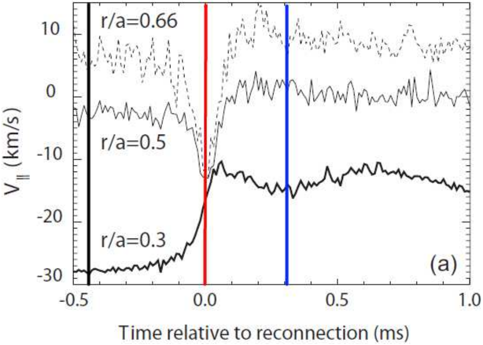


Den Hartog et. al. PoP **6** No. 5 (1999)

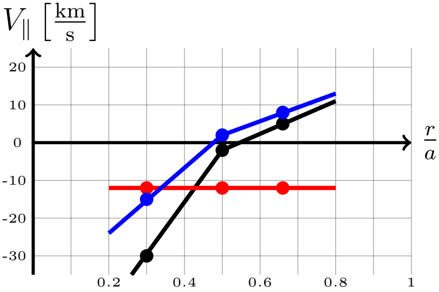
- Experimental sawtooth cycle in $F - \Theta$ space lies far from BFM
- Ohmic current drives plasma away from BFM, but sawtooth crashes drive plasma towards it

Intrinsic plasma flow is observed in MST and appears highly coupled with the relaxation dynamics.

- Significant parallel flow with strong shear between sawteeth
- Parallel flow flattens at crash suggesting **strong coupling between the flow and current relaxation**



Kuritsyn et. al. PoP **16** 055903 (2009)



- In the core, $dV_\parallel/dt > 0$ at the event

Plasma flow can be introduced into a variational formulation through the cross helicity.

- The **cross helicity** $\mathcal{X} \equiv \int \mathbf{v} \cdot \mathbf{B} \, d^3x$ is a measure of parallel flow[2]

$$\frac{\partial \mathcal{X}}{\partial t} = \int \left[\frac{1}{m_i n} \mathbf{F} \cdot \mathbf{B} - (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \boldsymbol{\nabla} \times \mathbf{v} \right] d^3x$$

- Invariant in **single-fluid ideal MHD** for: $\beta = 0$ <u>or</u> **barotropic** plasma

$$\frac{\mathbf{F} \cdot \mathbf{B}}{m_i n} = -\frac{\boldsymbol{\nabla} p}{m_i n} \cdot \mathbf{B} = -\frac{1}{m_i} \boldsymbol{\nabla} h \cdot \mathbf{B} = -\boldsymbol{\nabla} \cdot \left(\frac{h\mathbf{B}}{m_i} \right) \qquad \frac{dh}{dn} \equiv \frac{1}{n} \frac{dp}{dn}$$

- No reason to expect the cross helicity is better conserved than energy

$$\frac{\partial \mathcal{X}}{\partial t} \approx \int -\eta \mathbf{J} \cdot \boldsymbol{\nabla} \times \mathbf{v} \sim \int \eta \left[\sum_k k^2 v_k B_k \right] d^3x$$

- Variational principles that minimize energy while conserving magnetic helicity and cross helicity predict **field-aligned current and flow**

$$\delta \left[W_B + W_K - \frac{\lambda_0}{2\mu_0} \mathcal{K} - (m_i n) \lambda_1 \mathcal{X} \right] = 0 \rightarrow \left\{ \begin{array}{c} \mathbf{v} = \lambda_1 \mathbf{B} \\ \boldsymbol{\nabla} \times \mathbf{B} = \frac{\lambda_0}{1 - \mu_0 m_i n \lambda_1^2} \mathbf{B} \end{array} \right.$$

[2] Finn, J. M., T. J. Antonsen. 1983. Turbulent relaxation of compressible plasmas with flow. *Physics of Fluids* **26**(12) 3540–3552

An invariant hybrid helicity can be constructed if the Hall term is included in the generalized Ohm's law.[3]

- Hall physics in Ohm's law changes the cross helicity evolution

$$\frac{\partial \mathcal{X}}{\partial t} \sim \int -(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \boldsymbol{\nabla} \times \mathbf{v}\; d^3x \quad \rightarrow \quad \int \frac{1}{ne} \left(\boldsymbol{\nabla} p_e - \mathbf{J} \times \mathbf{B} \right) \cdot \boldsymbol{\nabla} \times \mathbf{v}\; d^3x$$

- Introduce **kinetic helicity**

$$\mathcal{H} \equiv \int \mathbf{v} \cdot \boldsymbol{\nabla} \times \mathbf{v}\; d^3x \qquad\qquad \frac{\partial \mathcal{H}}{\partial t} = \int \frac{2}{m_i n} \mathbf{F} \cdot \boldsymbol{\nabla} \times \mathbf{v}\; d^3x$$

- The **hybrid helicity** is a weighted sum of $\mathcal{K}$, $\mathcal{X}$, and $\mathcal{H}$

$$H \equiv \mathcal{K} + 2 \left(\frac{m_i}{e} \right) \mathcal{X} + \left(\frac{m_i}{e} \right)^2 \mathcal{H}$$

- The hybrid helicity is conserved in **ideal Hall MHD** if $p_i = 0$ or the plasma is barotropic

$$\frac{\partial H}{\partial t} = \int -\frac{2}{ne} \boldsymbol{\nabla} p_i \cdot \left[\mathbf{B} + \left(\frac{m_i}{e} \right) \boldsymbol{\nabla} \times \mathbf{v} \right]\; d^3x$$

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[3] Turner, L. 1986. Hall Effects on Magnetic Relaxation. *IEEE Transactions on Plasma Science* **PS-14**(6) 849–857

Conservation of hybrid helicity depends on coupling.

- Magnetic helicity with a two-fluid Ohm's law evolves as

$$\frac{\partial \mathcal{K}}{\partial t} = \int \left[\underline{-2\eta \mathbf{J}\cdot\mathbf{B}} + \underline{2\mathbf{B}\cdot\left(\tfrac{1}{ne}\boldsymbol{\nabla} p_e\right)} \right] d^3x$$

- The $\mathbf{B}\cdot\boldsymbol{\nabla} p_e$ from Hall physics couples this to cross helicity evolution

$$2\left(\frac{m_i}{e}\right)\frac{\partial \mathcal{X}}{\partial t} = \int \left[\underline{-2\left(\tfrac{m_i}{e}\right)\left(\tfrac{1}{ne}\right)(\mathbf{J}\times\mathbf{B})\cdot\boldsymbol{\nabla}\times\mathbf{v}} + \underline{2\left(\tfrac{m_i}{e}\right)\left(\tfrac{1}{ne}\right)(\boldsymbol{\nabla} p_e)\cdot\boldsymbol{\nabla}\times\mathbf{v}} \right] d^3x$$

$$+ \int \left[\underline{-2\left(\tfrac{m_i}{e}\right)\eta \mathbf{J}\cdot\boldsymbol{\nabla}\times\mathbf{v}} - \underline{2\mathbf{B}\cdot\left(\tfrac{1}{ne}\boldsymbol{\nabla} p_e\right)} - \underline{2\mathbf{B}\cdot\left(\tfrac{1}{ne}\boldsymbol{\nabla} p_i\right)} - \underline{2\mathbf{B}\cdot\left(\tfrac{1}{ne}\boldsymbol{\nabla}\cdot\underline{\boldsymbol{\Pi}}_i\right)} \right] d^3x$$

- The Hall terms in red and orange couple cross helicity and kinetic helicity

$$\left(\frac{m_i}{e}\right)^2\frac{\partial \mathcal{H}}{\partial t} = \int \left[\underline{2\left(\tfrac{m_i}{e}\right)\left(\tfrac{1}{ne}\right)\mathbf{J}\times\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{v}} - \underline{2\left(\tfrac{m_i}{e}\right)\left(\tfrac{1}{ne}\right)\boldsymbol{\nabla} p_e\cdot\boldsymbol{\nabla}\times\mathbf{v}} \right] d^3x$$

$$+ \int \left[\underline{-2\left(\tfrac{m_i}{e}\right)\left(\tfrac{1}{ne}\boldsymbol{\nabla} p_i\right)\cdot\boldsymbol{\nabla}\times\mathbf{v}} - \underline{2\left(\tfrac{m_i}{e}\right)\left(\tfrac{1}{ne}\boldsymbol{\nabla}\cdot\underline{\boldsymbol{\Pi}}_i\right)\cdot\boldsymbol{\nabla}\times\mathbf{v}} \right] d^3x$$

- Terms in yellow vanish for either cold ions or barotropic ions
- Terms in purple vanish for an ideal plasma
- The isotropic viscosity piece of the stress in green also vanishes for an ideal plasma, **but not the gyroviscous part**

The variational problem that minimizes energy while conserving hybrid helicity is singular in the limit $d_i \to 0$.

- Ignoring variations in density, the variational problem yields

$$\delta\left[W_B + W_K - \frac{\lambda}{2}H\right] = 0 \rightarrow \begin{cases} \nabla \times \mathbf{B} = \mu_0 \lambda \left[\mathbf{B} + \frac{m_i}{e}\nabla \times \mathbf{v}\right] \\ m_i n \mathbf{v} = \lambda \frac{m_i}{e}\left[\mathbf{B} + \frac{m_i}{e}\nabla \times \mathbf{v}\right] \end{cases}$$

- Ignoring the terms that are higher order in $\frac{m_i}{e} \sim d_i$ yields field-aligned currents and flows

$$\nabla \times \mathbf{B} \approx \mu_0 \lambda \mathbf{B} \qquad\qquad m_i n \mathbf{v} \approx \lambda \frac{m_i}{e}\mathbf{B}$$

- If the full equations are combined instead:

$$\left(\lambda d_i^2\right) \nabla \times \nabla \times \mathbf{B} = \nabla \times \mathbf{B} - \lambda \mathbf{B} \qquad\qquad \left(\frac{m_i}{e}\right)\mathbf{v} = d_i^2 \nabla \times \mathbf{B}$$

- The system is singular in the limit that $d_i \to 0$
- For $d_i \neq 0$, the value of λ must be chosen to satisfy initial conditions of the invariants (i.e. toroidal flux, magnetic helicity, hybrid helicity)[4]

[4] Khalzov, I. V., F. Ebrahimi, D. D. Schnack, V. V. Mirnov. 2012. Minimum energy states of the cylindrical plasma pinch in single-fluid and Hall magnetohydrodynamics. *Physics of Plasmas* **19**(012111)

It is well-known that the variational problem is mathematically ill-posed.[5]

- Consider minimizing F_1 subject to $u(0) = u(\pi) = 0$

$$F_1(u) = \int_0^{\pi} u^2 \, dx \qquad\qquad F_2(u) = \int_0^{\pi} \left(\frac{du}{dx}\right)^2 \, dx$$

- With no constraint, the minimum is simply $u = 0$
- Attempt to constrain this through conservation of F_2, a **fragile quantity**

$$0 = \delta\left[F_1 - \lambda F_2\right] = \int_0^{\pi} 2\delta u \left[u + \lambda \frac{d^2 u}{dx^2}\right] dx \rightarrow \begin{cases} u = C_o \sin(\alpha x) \\ \alpha \equiv 1/\sqrt{\lambda} = \pm 1, \pm 2, \pm 3, \ldots \end{cases}$$

- Use the solution in both F_1 and F_2. With F_2 invariant, C_0 is determined

$$F_1(u) = C_0^2 \frac{\pi}{2} \qquad\qquad F_2(u) = \alpha^2 C_0^2 \frac{\pi}{2} \qquad\qquad F_1 = F_2/\alpha^2$$

- The minimum is $\alpha^2 \rightarrow \infty$ or $\lambda \rightarrow 0$, i.e. $0 = \delta\left[F_1 - \lambda F_2\right] \rightarrow 0 = \delta F_1$

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[5] Ohsaki, S., Z. Yoshida. 2005. Variational principle with singular perturbation of Hall magnetohydrodynamics.
Physics of Plasmas **12**(064505)

NIMROD, a 3D extended MHD code that includes two-fluid physics, is used to study relaxation dynamics.

- The model includes two-fluid physics and first order FLR corrections:

$$\frac{\partial n}{\partial t} = -\nabla \cdot (n\mathbf{v}) + D_n \nabla^2 n$$

$$m_i n \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \mathbf{J} \times \mathbf{B} - \nabla (nT) - \nabla \cdot \underline{\mathbf{\Pi}}_{iso} - \underline{\nabla \cdot \underline{\mathbf{\Pi}}_{gv}}$$

$$\frac{n}{\Gamma - 1} \left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = -nT (\nabla \cdot \mathbf{v}) + \nabla \cdot (\chi n \nabla T)$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \left[-\mathbf{v} \times \mathbf{B} + \underline{\tfrac{1}{ne} (\mathbf{J} \times \mathbf{B} - \nabla p_e)} + \eta \mathbf{J} + \underline{\tfrac{m_e}{ne^2} \tfrac{\partial \mathbf{J}}{\partial t}} \right]$$

$$\underline{\mathbf{\Pi}}_{iso} = \nu m_i n \underline{\mathbf{W}} \qquad \underline{\mathbf{W}} = \nabla \mathbf{v} + \nabla \mathbf{v}^T - \frac{2}{3} (\nabla \cdot \mathbf{v}) \underline{\mathbf{I}}$$

$$\underline{\mathbf{\Pi}}_{gv} = \frac{m_i p_i}{4eB} \left[\hat{\mathbf{b}} \times \underline{\mathbf{W}} \cdot \left(\underline{\mathbf{I}} + 3\hat{\mathbf{b}}\hat{\mathbf{b}} \right) - \left(\underline{\mathbf{I}} + 3\hat{\mathbf{b}}\hat{\mathbf{b}} \right) \cdot \underline{\mathbf{W}} \times \hat{\mathbf{b}} \right]$$

- Simulation Parameters:

$$S = 20,000 \qquad P_m = \nu/\eta = 1 \qquad \chi/\eta = 0.1 = D_n/\eta \qquad \tau_A = 1$$

$$\beta = 0.1 \qquad d_i/a = 0.173 \qquad \rho_s/a = 0.05 \qquad m_e/m_i = 2.72 \cdot 10^{-3}$$

- **MHD** all not underlined, **Cold Ion** + red, **Warm Ion** + red & blue

Diagnostics examine which terms are responsible for changes in magnetic energy and helicity in simulations.

- NIMROD[6] represents fields with $\mathcal{C}^0$ finite elements in the $R-Z$ plane and a Fourier series in ϕ
- Integration-by-parts is used to eliminate second derivatives:

$$\int 2T\boldsymbol{\nabla}\cdot\left(D_n\boldsymbol{\nabla}n\right)d^3x = \int\left[\boldsymbol{\nabla}\cdot\left(2TD_n\boldsymbol{\nabla}n\right) - \left(D_n\boldsymbol{\nabla}n\right)\cdot\boldsymbol{\nabla}\left(2T\right)\right]d^3x$$

$$\int \mathbf{v}\cdot\left(\boldsymbol{\nabla}\cdot\underline{\boldsymbol{\Pi}}\right)d^3x = \int\left[\boldsymbol{\nabla}\cdot\left(\underline{\boldsymbol{\Pi}}\cdot\mathbf{v}\right) - \underline{\boldsymbol{\Pi}}:\left(\boldsymbol{\nabla}\mathbf{v}\right)^T\right]d^3x$$

- Auxiliary fields are required for higher-order derivatives

$$\int \boldsymbol{\alpha}\cdot\left(\boldsymbol{\nabla}\cdot\underline{\boldsymbol{\Pi}}\right)d^3x = \int\left\{\boldsymbol{\nabla}\cdot\left[\underline{\boldsymbol{\Pi}}\cdot\boldsymbol{\alpha}\right] - \underline{\underline{\boldsymbol{\Pi}}:\left[\boldsymbol{\nabla}\boldsymbol{\alpha}\right]^T}\right\}d^3x \qquad \alpha = \boldsymbol{\nabla}\times\mathbf{v}$$

- Terms are constructed by transforming to real space and aliasing errors may be present for combinations of more than two fields

$$\int \frac{1}{ne}\left(\mathbf{J}\times\mathbf{B}\right)\cdot\boldsymbol{\nabla}\times\mathbf{v}\ d^3x$$

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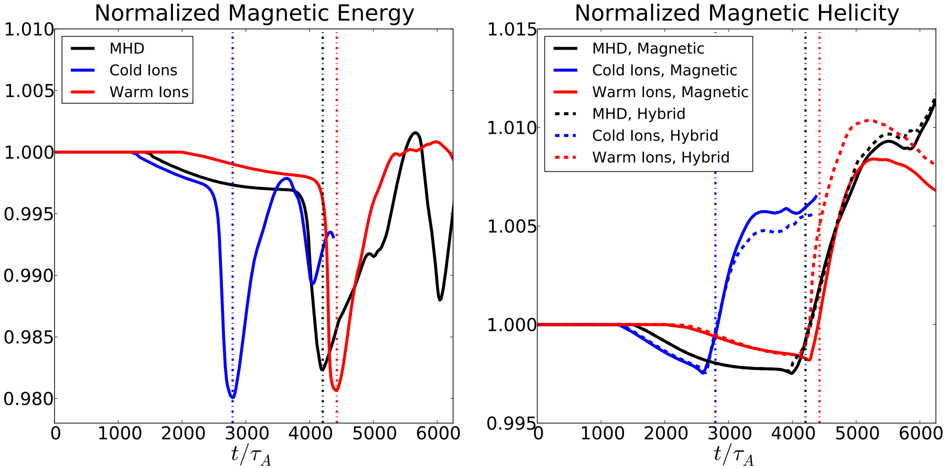
[6] Sovinec, C. R. et. al. 2004. Nonlinear magnetohydrodynamics simulation using high-order finite elements.
Journal of Computational Physics **195** 355–386

The magnetic helicity is more robustly conserved than the magnetic energy at the relaxation event.



- Dashed vertical lines indicate the time of minimum magnetic energy
- The magnetic helicity has changed by only a small percentage while the energy has dropped by $\sim 2\%$
- The hybrid helicity is very nearly equal to the magnetic helicity

The constant loop voltage injects magnetic energy and helicity into the system.

- The magnetic energy evolves as

$$\frac{\partial W_B}{\partial t} = -\int \frac{\mathbf{E}\times\mathbf{B}}{\mu_0}\cdot\hat{\mathbf{n}}dA + \int \left[\mathbf{v}\times\mathbf{B}\cdot\mathbf{J} - \eta J^2 + \mathbf{J}\cdot\frac{1}{ne}\boldsymbol{\nabla}p_e\right] d^3x$$

- The only tangential electric field is the constant loop voltage, which balances the contributions from the equilibrium $\mathbf{E}_{\mathrm{eq}} = -\mathbf{v}_{\mathrm{eq}}\times\mathbf{B}_{\mathrm{eq}} + \eta\mathbf{J}_{\mathrm{eq}}$

$$\frac{\partial W_B}{\partial t} = \int \left[\underline{(\mathbf{v}\times\mathbf{B} - \mathbf{v}_{\mathrm{eq}}\times\mathbf{B}_{\mathrm{eq}})\cdot\mathbf{J}} - \underline{\eta\left(\mathbf{J}-\mathbf{J}_{\mathrm{eq}}\right)\cdot\mathbf{J}} + \underline{\mathbf{J}\cdot\tfrac{1}{ne}\boldsymbol{\nabla}p_e}\right] d^3x$$

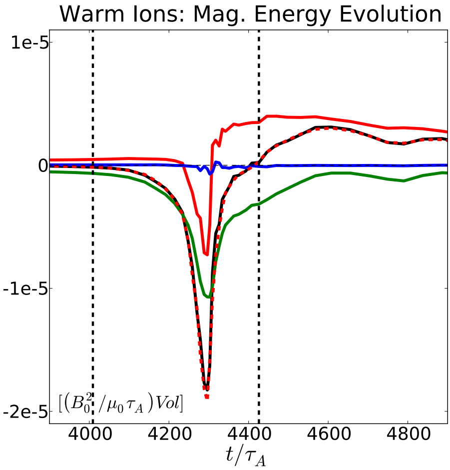
- The relative magnetic helicity $\mathcal{K}_{rel} \equiv \int (\mathbf{A}-\mathbf{A}')\cdot(\mathbf{B}+\mathbf{B}')\, d^3x$ evolves as

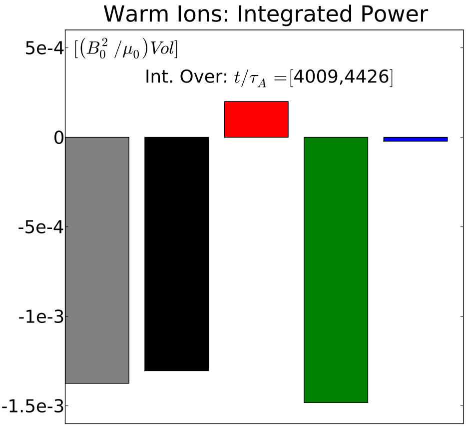
$$\frac{\partial \mathcal{K}}{\partial t} = -2\int \left[\mathbf{E}\cdot\mathbf{B} - \mathbf{E}'\cdot\mathbf{B}'\right] d^3x = -2\int \left[\eta\mathbf{J}\cdot\mathbf{B} - \mathbf{B}\cdot\frac{1}{ne}\boldsymbol{\nabla}p_e - \mathbf{E}'\cdot\mathbf{B}'\right] d^3x$$

- The reference fields must have the same tangential electric field and total magnetic flux so that the relative helicity evolution is

$$\frac{\partial \mathcal{K}}{\partial t} = \int \left[-\underline{2\left(\mathbf{v}_{\mathrm{eq}}\times\mathbf{B}_{\mathrm{eq}}\right)\cdot\mathbf{B}} - \underline{2\eta\left(\mathbf{J}-\mathbf{J}_{\mathrm{eq}}\right)\cdot\mathbf{B}} + \underline{2\mathbf{B}\cdot\tfrac{1}{ne}\boldsymbol{\nabla}p_e}\right] d^3x$$

The magnetic energy evolution over the crash is dominated by the resistive term.

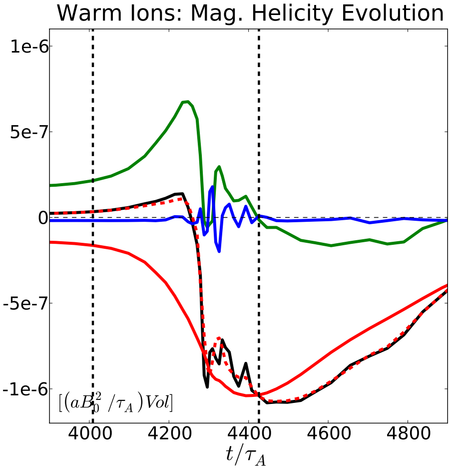


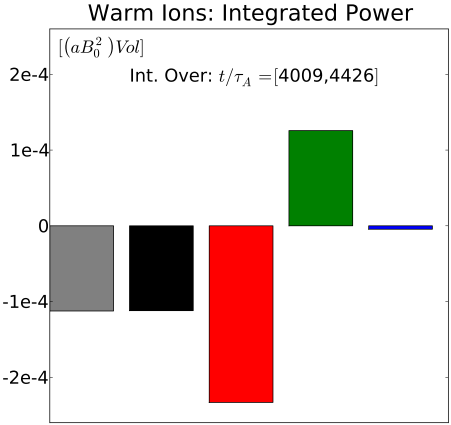


- The dashed red curve shows a finite difference estimate of $\frac{\partial W_B}{\partial t}$

$$\frac{\partial W_B}{\partial t} = \int \left[\underline{(\mathbf{v}\times\mathbf{B} - \mathbf{v}_{\mathrm{eq}}\times\mathbf{B}_{\mathrm{eq}})\cdot\mathbf{J}} - \underline{\eta\,(\mathbf{J}-\mathbf{J}_{\mathrm{eq}})\cdot\mathbf{J}} + \underline{\mathbf{J}\cdot\tfrac{1}{ne}\boldsymbol{\nabla} p_e} \right] d^3x$$

The electron pressure in magnetic helicity evolution is weak and there is little coupling of $\mathcal{K}$ and $\mathcal{X}$.

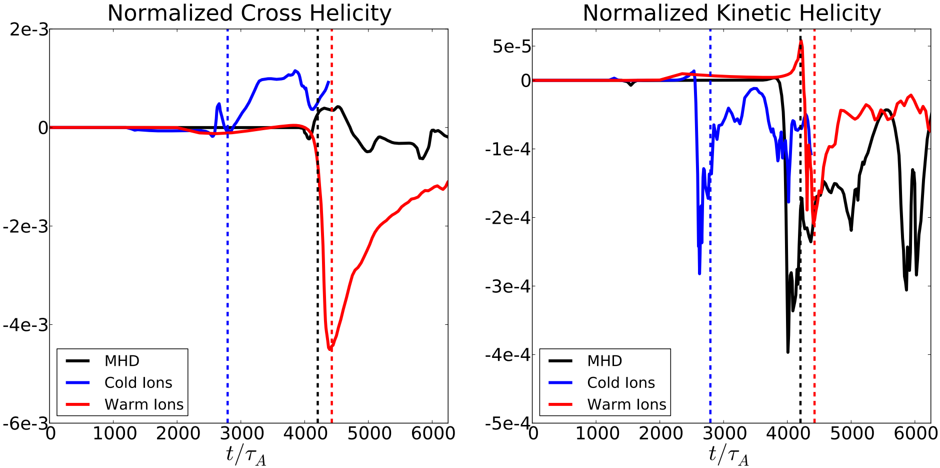




- The dashed red curve shows a finite difference estimate of $\frac{\partial \mathcal{K}}{\partial t}$

$$\frac{\partial \mathcal{K}}{\partial t} = \int \left[-\underline{2\left(\mathbf{v}_{\mathrm{eq}} \times \mathbf{B}_{\mathrm{eq}}\right) \cdot \mathbf{B}} - \underline{2\eta\left(\mathbf{J} - \mathbf{J}_{\mathrm{eq}}\right) \cdot \mathbf{B}} + \underline{2\mathbf{B} \cdot \frac{1}{ne} \boldsymbol{\nabla} p_e} \right] d^3x$$

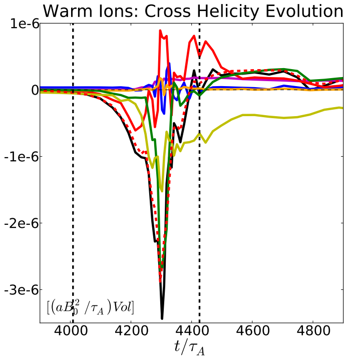
Including the ion gyroviscosity significantly alters the evolution of the cross helicity at the relaxation event.

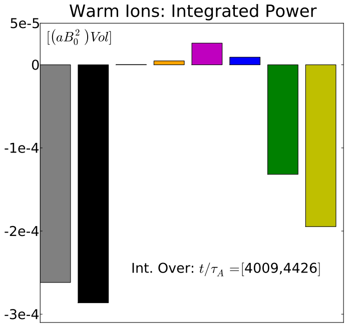


- The cross and kinetic helicities are normalized by the initial value of the magnetic helicity
- The kinetic helicity evolution appears similar for all cases

Cross helicity evolution is dominated by the viscous and gyroviscous pieces in simulations.

- Variational theories that conserve hybrid helicities neglect viscous dissipation and do not account for gyroviscous effects





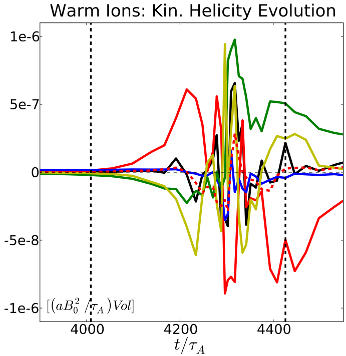
$$2\left(\frac{m_i}{e}\right)\frac{\partial \mathcal{X}}{\partial t} = \int \left[\underline{-2\left(\frac{m_i}{e}\right)\left(\frac{1}{ne}\right)(\mathbf{J}\times\mathbf{B})\cdot\nabla\times\mathbf{v}} + \underline{2\left(\frac{m_i}{e}\right)\left(\frac{1}{ne}\right)\nabla p_e\cdot\nabla\times\mathbf{v}}\right]d^3x$$

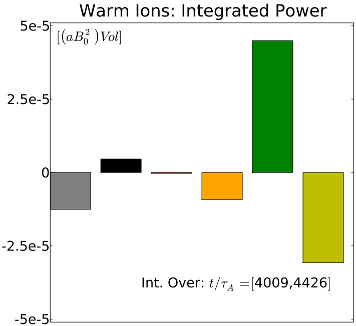
$$+\int\left[\underline{-2\left(\frac{m_i}{e}\right)\eta\mathbf{J}\cdot\nabla\times\mathbf{v}} - \underline{2\mathbf{B}\cdot\left(\frac{1}{ne}\nabla p\right)} - \underline{\frac{2\mathbf{B}}{ne}\cdot(\nabla\cdot\underline{\mathbf{\Pi}}_{iso})} - \underline{\frac{2\mathbf{B}}{ne}\cdot\left(\nabla\cdot\underline{\mathbf{\Pi}}_{gyr}\right)}\right]d^3x$$

The kinetic helicity evolution is under-resolved but has large contributions from viscosity and gyroviscosity.

- The kinetic helicity contribution is small: $\mathcal{H} \sim 10^{-1}\mathcal{X}$ and $\mathcal{H} \sim 10^{-4}\mathcal{K}$

$$H = \left[aB_0^2 Vol\right] \int \left\{ \hat{\mathbf{A}} \cdot \hat{\mathbf{B}} + 2\left(\frac{d_i}{a}\right) \hat{\mathbf{v}} \cdot \hat{\mathbf{B}} + \left(\frac{d_i}{a}\right)^2 \hat{\mathbf{v}} \cdot \left(\hat{\boldsymbol{\nabla}} \times \hat{\mathbf{v}}\right) \right\} d^3\hat{x}$$





$$\left(\frac{m_i}{e}\right)^2 \frac{\partial \mathcal{H}}{\partial t} = 2\frac{m_i}{e} \int \left[\underline{\mathbf{J} \times \mathbf{B}} - \underline{\boldsymbol{\nabla} p} - \underline{\boldsymbol{\nabla} \cdot \underline{\mathbf{\Pi}}_{iso}} - \underline{\boldsymbol{\nabla} \cdot \underline{\mathbf{\Pi}}_{gyr}} \right] \cdot \frac{\boldsymbol{\nabla} \times \mathbf{v}}{ne}\, d^3x$$

Conclusions

- Relaxation theories attempt to predict the preferred plasma state by minimize some quantities while conserving others
  - Ideal invariants in a two-fluid model are the energy and the hybrid helicity
  - Minimizing the energy while conserving the hybrid helicity results in an ill-posed mathematical problem (see Ohsaki ref., slide 10)
- Numerical simulations examine the evolution of the ideal invariants within the extended MHD model
  - Magnetic helicity is robustly conserved relative to magnetic energy
  - Cross helicity evolution is dominated by viscosity and gyroviscosity
  - First order FLR effects (ion gyroviscosity) has **not** been included in any relaxation theories but warm ion simulations suggest it is important
- Diagnostics measuring helicity evolution accurately capture the large scale dynamics
  - Kinetic helicity evolution appears under-resolved
  - However, it is the smallest contributor to hybrid helicity
  - To the order of the cross helicity, the evolution is well-resolved