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Formation and Instability of Phase Space Structures

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Outline

- **Formation of nonlinear phase-space structures.**
- **Adiabatic evolution of the structures.**
- **Stability analysis of the structures.**
- **Mutual interaction between multiple structures.**

Nonlinear Evolution of Waves due to Weak Instabilities

- **Resonant particles destabilize waves a plasma can support.**

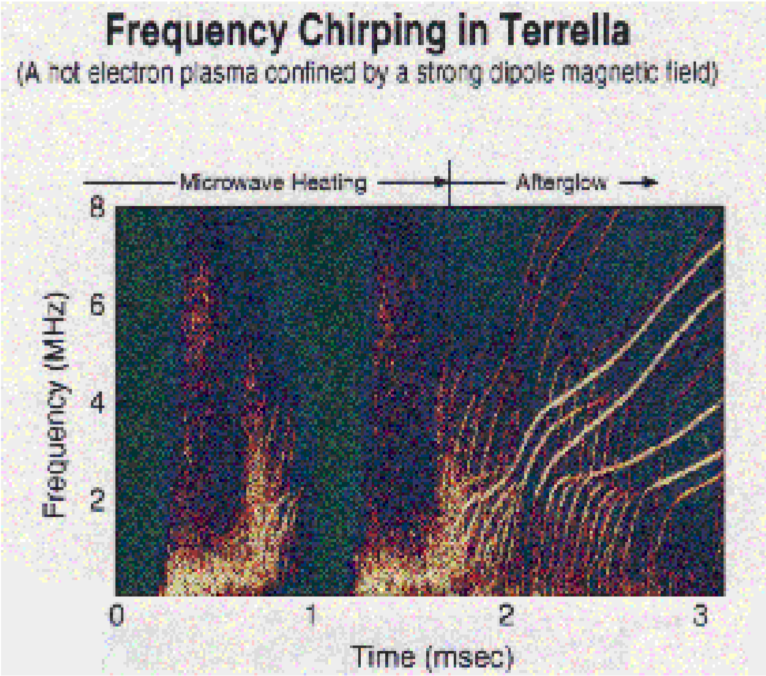
- **System is close to the linear instability threshold, where kinetic drive = background dissipation.**

- **Spatial structure of waves remains unchanged.**

- **Dominant nonlinearity is caused by interaction with resonant particles**

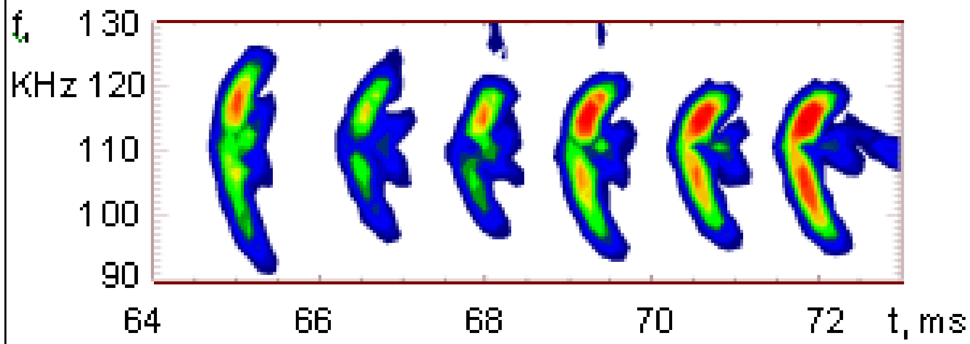
Formation of Phase Space Structures (Experimental Data)

Terrella experiment



Columbia University, Mauel et. al.

TAE modes in MAST



Culham Laboratory, U. K. courtesy of Mikhail Gryaznevich

Signature for Formation of Phase Space Structures (Theory)

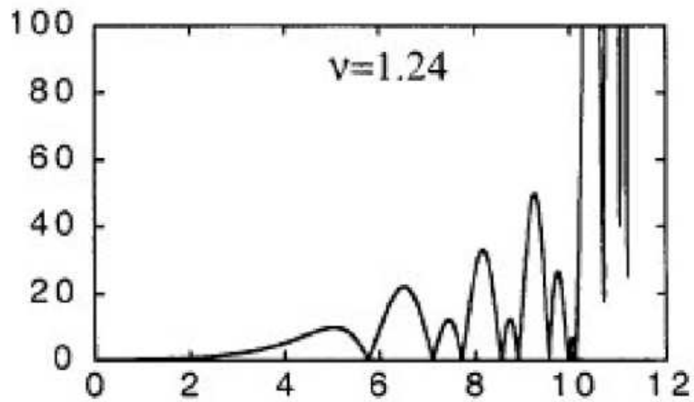
$\gamma_L - \gamma_d \ll \gamma_L$ **[Berk, Breizman, and Pekker, Plasma Phys. Rep. (1997)]**

$$\boxed{\frac{d\tilde{A}}{dt} = \tilde{A} - \frac{1}{2}\int_0^{t/2} d\tau\,\tau^2 \int_0^{t-2\tau} d\tau_1 G(\tau,\tau_1)\tilde{A}(t-\tau)\tilde{A}(t-\tau-\tau_1)\tilde{A}^*(t-2\tau-\tau_1)}$$

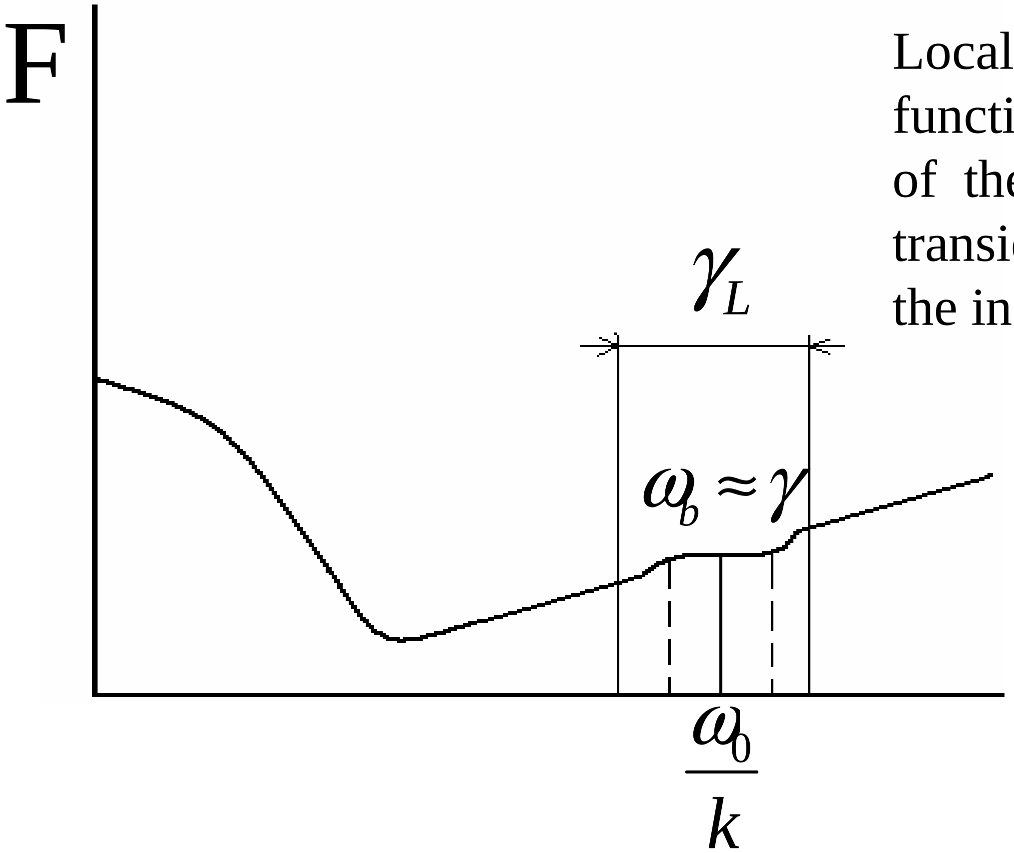
Explosive self-similar solution for small collisional effects

$$A(t) = \frac{1}{2}\sum_{n=0}^{\infty} g_n \{\exp[i\sigma \ln(t_0 - t)] + c.c\}/(t_0 - t)^{5/2}$$

Numerical simulations by M.S. Pekker: the solution blows up in finite time



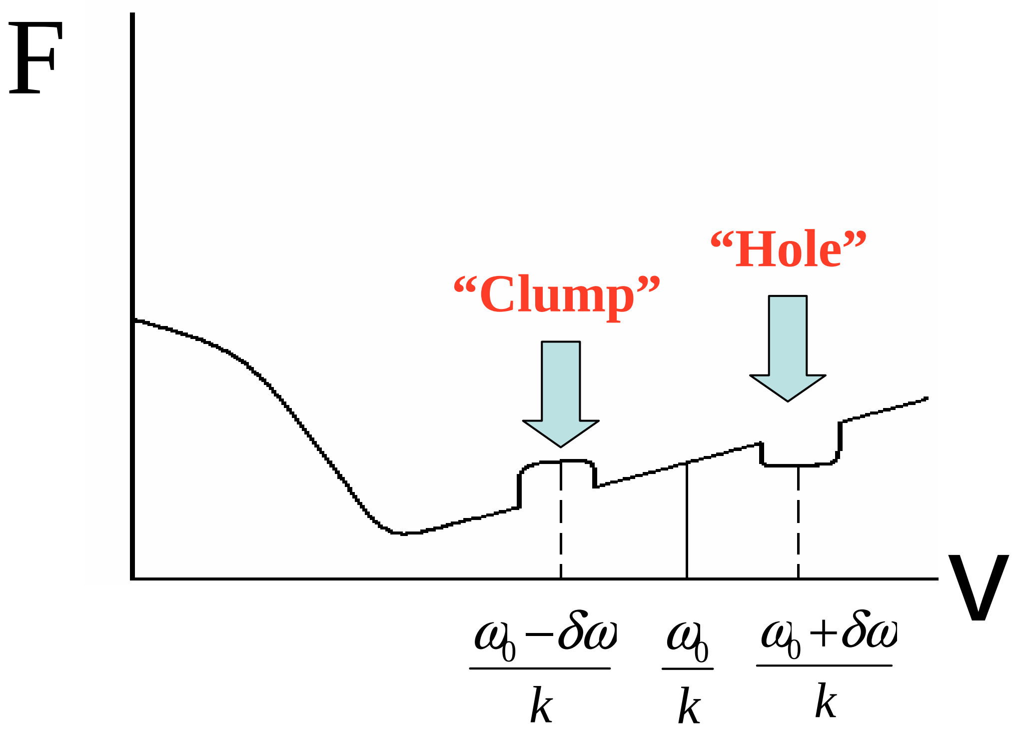
Explosive Solution



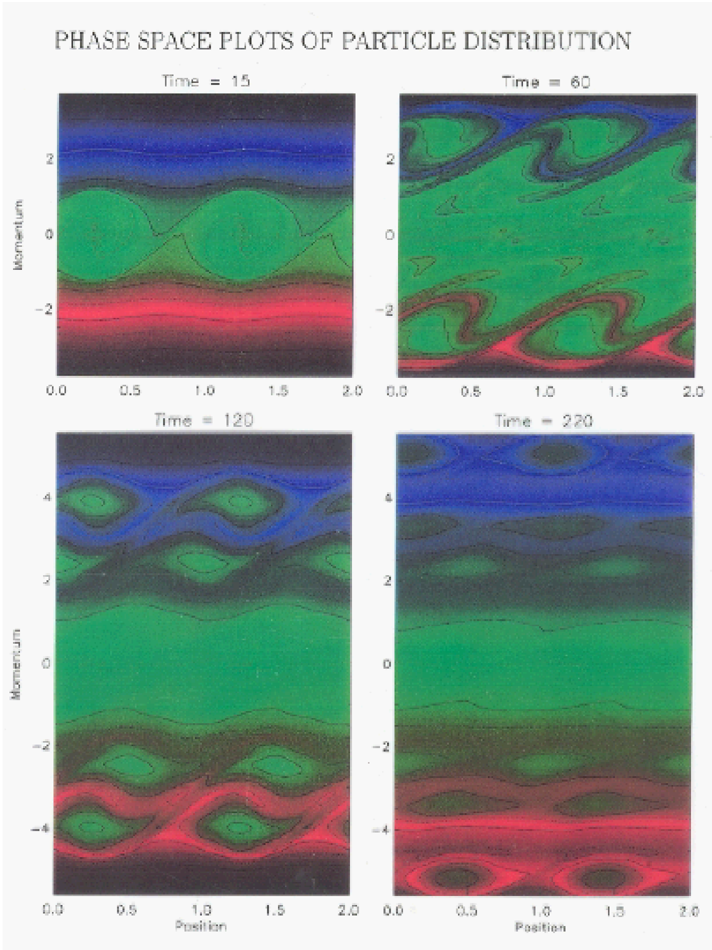
Local flattening of the distribution function produces large gradients of the distribution function in the transient regions. The gradients feed the instability further.

$$\omega_b \sim \sqrt{A}$$

Formation of "Holes" and "Clumps"



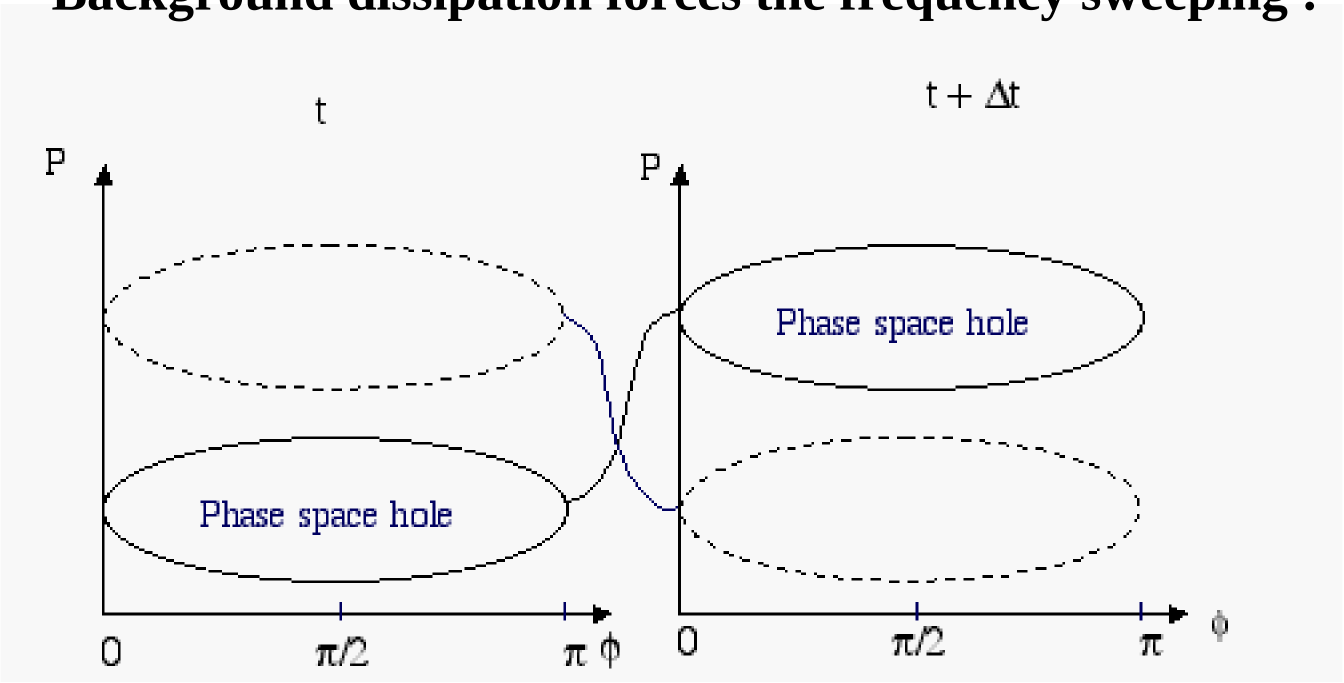
Simulation: N. Petviashvili



Formation of Nonlinear Phase Space Structures

Frequency Sweeping of Nonlinear Phase Space Structures

Background dissipation forces the frequency sweeping !



Slow Evolution of the Structures

Slowly changing wave properties: $\max\left(\frac{d\delta\omega}{dt}, \frac{d\omega_b}{dt}\right) \ll \omega_b^2$

Adiabatic invariants for description of the evolution of the trapped particle distribution function

$$J = \frac{1}{2\pi}\oint d\xi p_\xi = \frac{1}{\pi}\int_{-\xi_{\max}}^{\xi_{\max}} \left[2(E+\omega_b^2\cos(\xi))\right]^{1/2} = const \quad \Rightarrow F(J) = const$$

+ equations for the mode phase and amplitude

Self-consistent adiabatic analysis,
stable evolution of waves and particles

Description of Adiabatic Evolution

Evolution of the mode's normalized trapping frequency Ω_b in terms of normalized frequency shift Ω is governed by

$$\boxed{\Omega_b = 1 - \frac{1}{\Omega}\int_0^1 dI\, Q(I)\, \mathrm{F}_T(\Omega_b I)}$$

$$J \sim \Omega_b I$$

$$0 \le I \le 1$$

$$Q(I) = 3\frac{\int_0^{\xi_{\max}} d\xi \cos\xi / [E + \cos\xi]^{1/2}}{\int_0^{\xi_{\max}} d\xi / [E + \cos\xi]^{1/2}}$$

Ω normalized frequency shift

Ω_b normalized trapping frequency

Problems in Adiabatic Description

Differentiating $\Omega_b = 1 - \frac{1}{\Omega}\int_0^1 dI Q(I) f_T(\Omega_b I)$, one obtains:

$$\boxed{\frac{d\Omega}{d\Omega_b} = \frac{\Omega H_T(\Omega, \Omega_b)}{1-\Omega_b}}$$

$$H_T(\Omega, \Omega_b) = 1 + \frac{1}{\Omega}\int_0^1 dI f_T{}'(\Omega_b I) I Q(I)$$

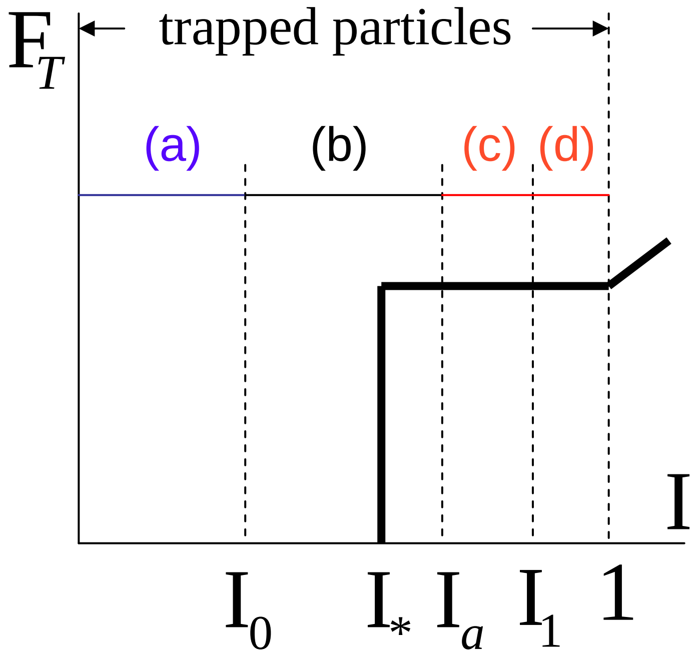
Frequency termination point:

$$H_T \to 0$$

Bifurcation point:

$$H_T \to 0 \quad \Omega_b \to 1$$

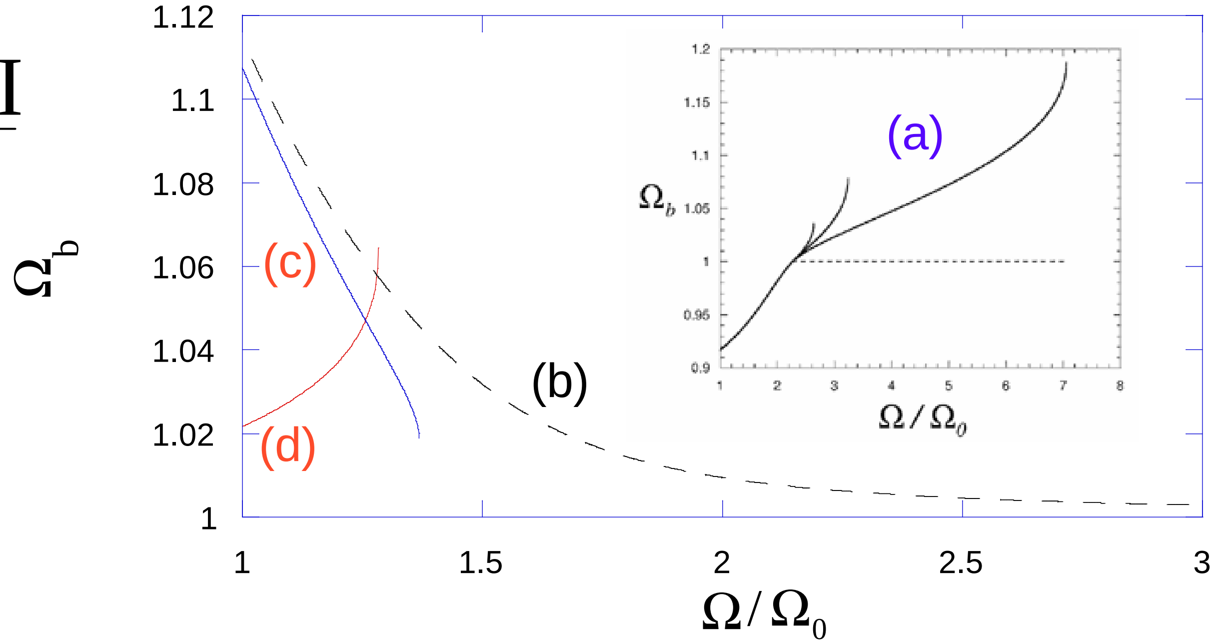
Four Regimes of Solutions



Initial trapped particle distribution function:

$$F_{T0}\left(I;I_*\right)=\Omega_0\times\theta\left(I-I_*\right)$$

Trapping Frequency vs Frequency Shift



Adiabatic Evolution of the Structures

Why does the adiabatic theory fail?

Self-Consistent Perturbative Analysis

Perturbed equations for the mode's phase and amplitude

and

Linearized Vlasov equation

$$\frac{D\delta f}{Dt} = \dot{\xi}\frac{\partial \delta\varphi}{\partial \xi}\frac{\partial F(E)}{\partial E} = \frac{\partial F(E)}{\partial E}\left(\frac{D}{Dt} - \frac{\partial}{\partial t}\right)\delta\varphi \qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + \dot{\xi}\frac{\partial}{\partial \xi} - \frac{\partial H_0}{\partial \xi}\frac{\partial}{\partial \dot{\xi}}$$

Perturbation of the distribution function:

$$\delta f = \frac{\partial F(E)}{\partial E}\left[\delta\varphi(\xi,t) - \int_{-\infty}^{t} dt'\frac{\partial}{\partial t'}\delta\varphi(\xi,t')\right]$$

Integration along unperturbed trajectories

$$\delta\varphi(\xi,t) \approx -\Delta A\cos(\xi) - \Delta\alpha A_0\sin(\xi)$$

[Eremin, Berk, Phys. Plasmas (2004)]

Dispersion Relation for the Perturbed Eigenmodes

$$\boxed{\omega^2 D_2\left(\omega^2\right) = D_1 + \frac{A_0}{\Delta\omega D_3\left(\omega^2\right)}}$$

$$D_1 = \Delta\omega\left[1 - \beta\int_{-A_0}^{A_0} dE \frac{1}{\Delta\omega}\frac{\partial G(E)}{\partial E} T(E)\left[\left\langle \cos^2(\xi)\right\rangle_0^c - \left(\left\langle \cos(\xi)\right\rangle_0^c\right)^2\right]\right]$$

$$D_2 = 2\beta\int_{-A_0}^{A_0} dE \frac{\partial G(E)}{\partial E} T(E)\sum_{p=1}^{\infty}\left[\frac{\left(\left\langle \cos(\xi)\right\rangle_{2p}^c\right)^2}{\left(2p\omega_b(E)\right)^2 - \omega^2}\right]$$

$$D_3 = 2\beta A_0\int_{-A_0}^{A_0} dE \frac{1}{\Delta\omega}\frac{\partial G(E)}{\partial E} T(E)\sum_{p=1}^{\infty}\left[\frac{\left(\left\langle \sin(\xi)\right\rangle_{2p-1}^c\right)^2}{\left((2p-1)\omega_b(E)\right)^2 - \omega^2}\right]$$

Instability Analysis

$A_0^{1/2}/\Delta\omega \ll 1$ $\boxed{\gamma^2 D_2\left(-\gamma^2\right) = -D_1}$ $\gamma \equiv -i\omega$

If $\dfrac{\partial G}{\partial E} > 0$ **then lhs is positive,** $\gamma^2 D_2\left(-\gamma^2\right) = 2\beta\gamma^2 \int_{-A_0}^{A_0} dE \dfrac{\partial G(E)}{\partial E} T(E) \sum_{p=1}^{\infty} \left[\dfrac{\left(\left\langle \cos(\xi) \right\rangle_{2p}^{c}\right)^2}{\left(2p\omega_b(E)\right)^2 + \gamma^2} \right] > 0$

$$\gamma^2 D_2\left(-\gamma^2\right) \xrightarrow[\gamma\to\infty]{} 2\beta \int_{-A_0}^{A_0} dE \frac{\partial G(E)}{\partial E} T(E) \sum_{p=1}^{\infty} \left(\left\langle \cos(\xi) \right\rangle_{2p}^{c}\right)^2 \equiv D > 0$$

Unstable root guaranteed if $D_1 < 0$

After a great deal of algebraic manipulations, we find that $D_1 = \dfrac{H_T}{2}$

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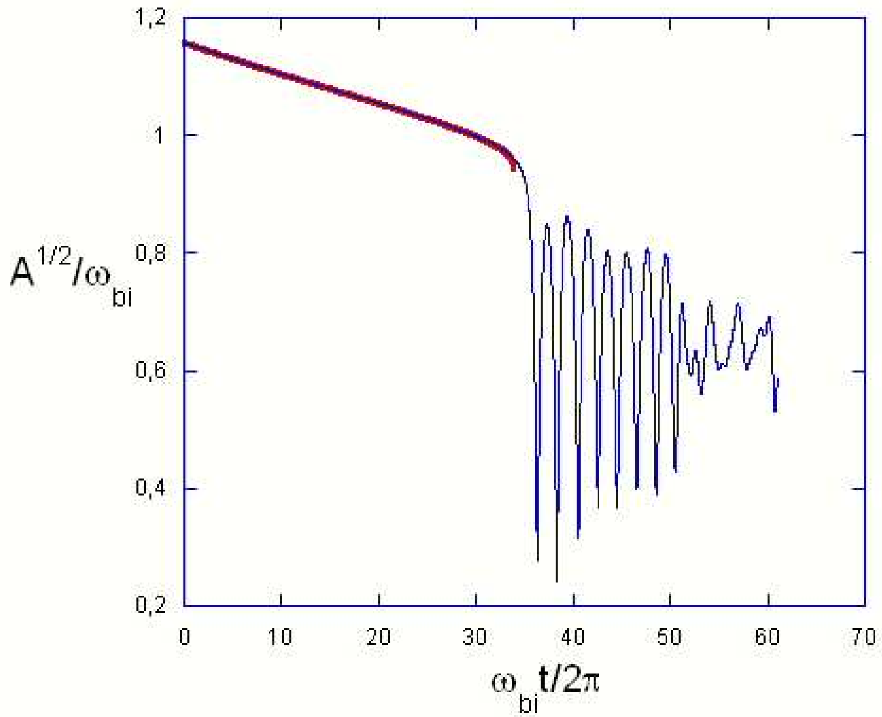
After a great deal of algebraic manipulations, we find that $D_1 = \dfrac{H_T}{2}$

$\Rightarrow$ Adiabatic theory "knows" about onset of the instability!

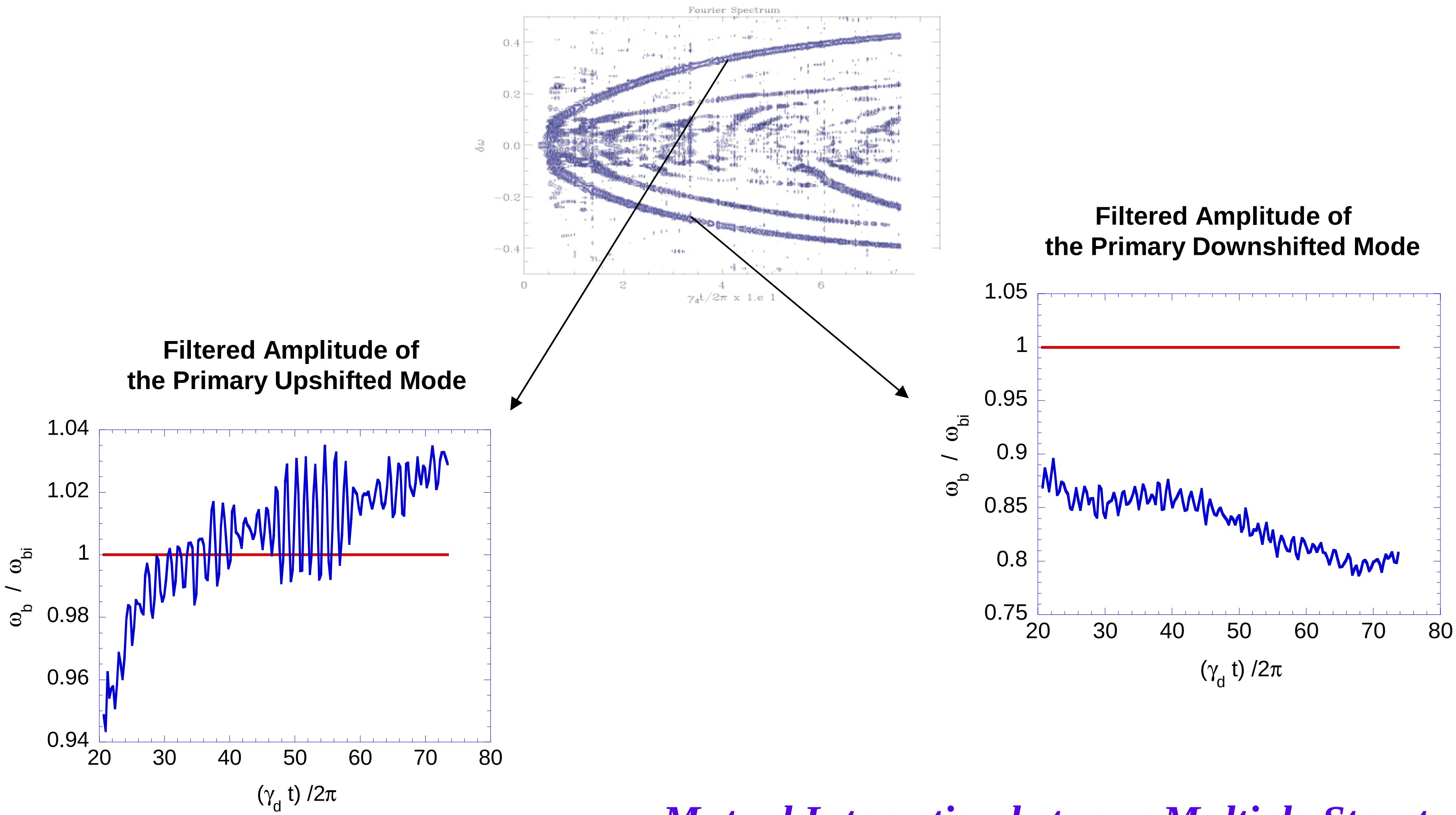
Numerical Results: Dynamic Run



Comparison of predicted evolution with particle simulation for a case when a frequency termination point is reached.

$$\omega_{bi} = 0.011, \gamma_d / \omega_{bi} = 0.37, I_* = 0.8$$
$$\Delta I = 0.02, \Delta\omega_0 / \omega_{bi} = 18.5$$

Self-Consistent Dynamics of Multiple Structures



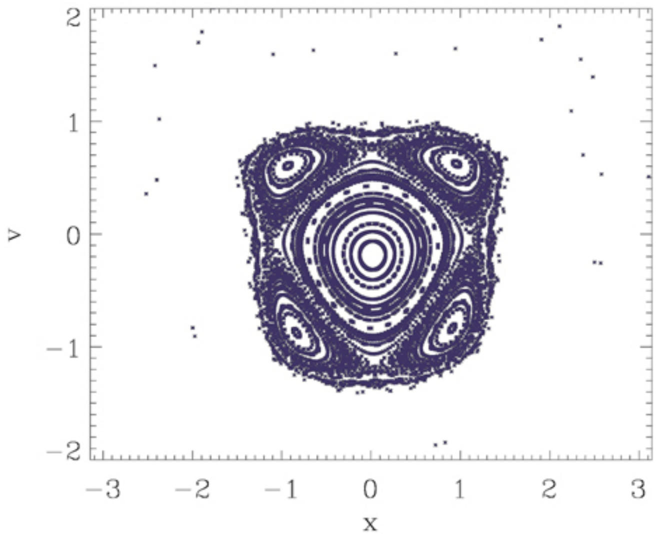
Mutual Interaction between Multiple Structures

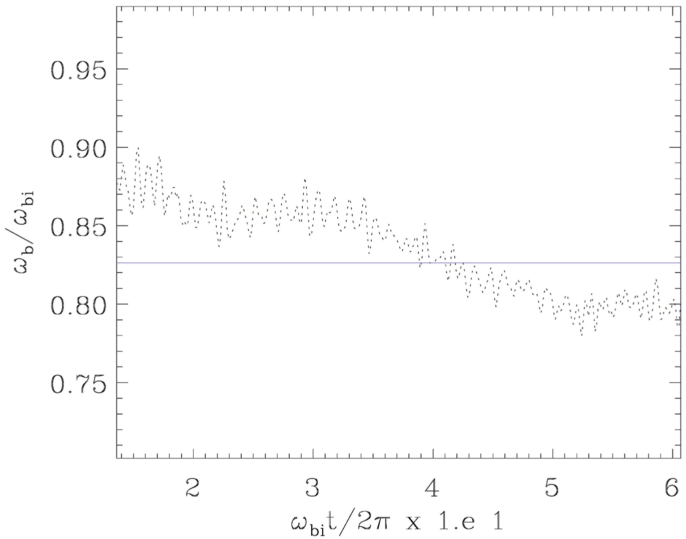
Mechanisms for Amplitude Reduction

Regular Dynamics: contribution from the trapped region of the neighboring mode

Chaotic Erosion of the Separatrix: stochastic instability + frequency sweeping

Explain observed amplitude!





$$H = \frac{v_\xi^2}{2} - \omega_{b1}^2\left(\cos(\xi) - \varepsilon_\omega \xi\right) - \omega_{b2}^2 \cos(\xi - \Delta\omega t)$$

Mutual Interaction between Multiple Structures

Summary

- **Nonlinear phase space structures occur spontaneously in a resonant system with damping, close to instability threshold.**

- **The adiabatic analysis very accurately describes the frequency sweeping when the mode is stable.**

- **The self-consistent adiabatic solution may evolve to points, where the adiabatic analysis fails.**

- **Linear perturbative analysis demonstrates that these points are exactly where an instability is triggered.**

Summary (cont.)

- **Nonlinear response interesting, needs further study.**

- **Generation of subsidiary structures changes mode amplitude; analytic prediction successfully made.**

- **Theoretical arguments considered here should be important in understanding the experimental data.**