



Astrophysical Dynamo Action

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Preliminary considerations

A hydromagnetic dynamo is a sustained mechanism to convert kinetic energy into magnetic energy within the bulk of an electrically conducting fluid

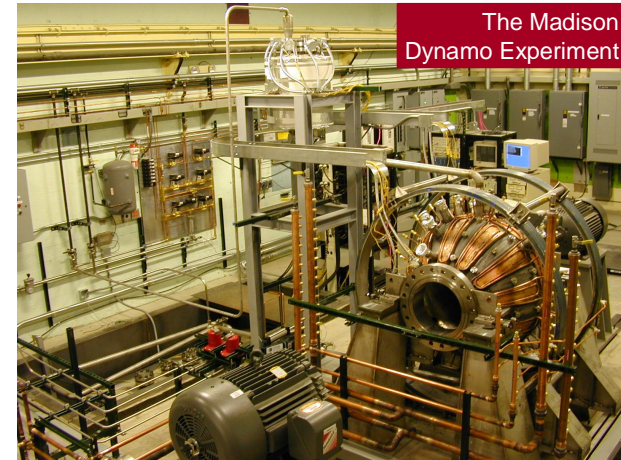
- **Commonly invoked to explain the origin of cosmic magnetic fields**
- **Universe is a strongly magnetized place (now)**
- **WMAP. Universe was not strongly magnetized at recombination**
- **Something must have generated magnetic fields between then and now**

Assume magnetic fields are generated by dynamos

Some examples

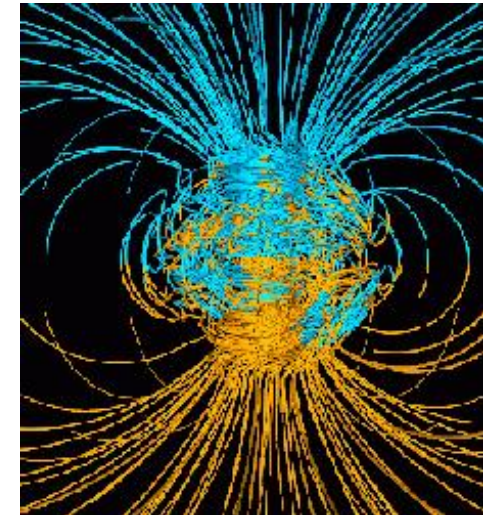
Liquid metal experiments

- Typical size 0.5 – 2.0 m
- Turbulence driven by propellers



Geodynamo

- Size: 6,400 Km
- Turbulence driven by compositional convection in the liquid core
- Evidence for magnetic reversals

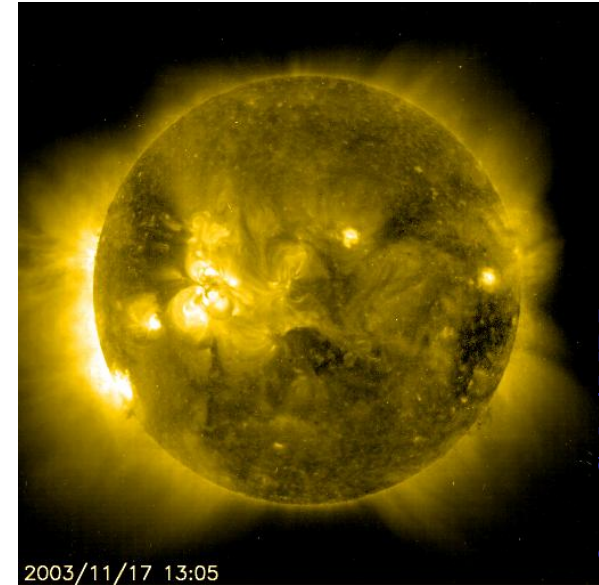


from Glatzmaier

Some examples

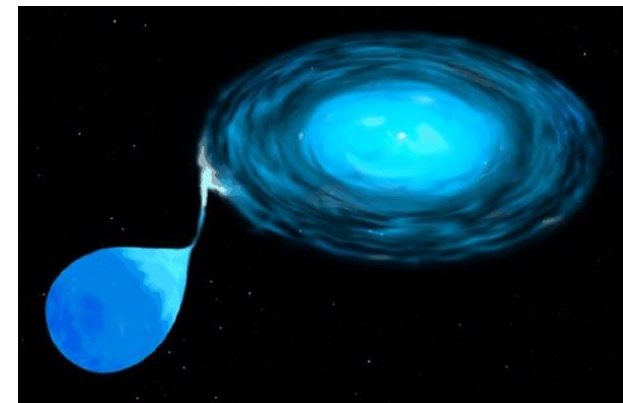
Sun (late-type stars)

- Size 600,000 Km
- Turbulence driven by thermal convection
- Evidence for activity cycles



Accretion disks

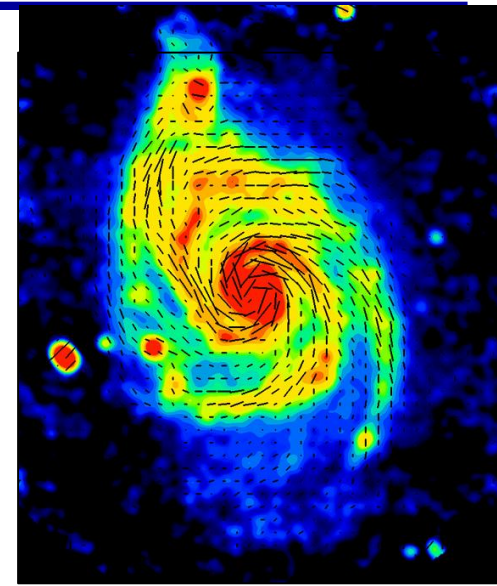
- Typical size varies
- Turbulence driven by MRI



Some examples

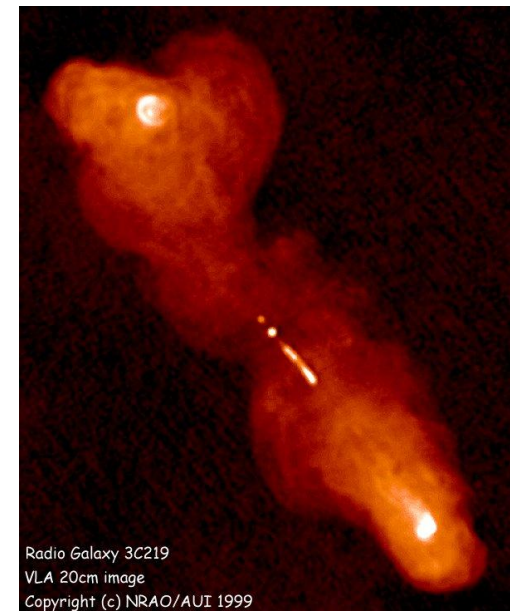
Galaxy

- Typical size: 10^{20} m
- Turbulence driven by supernovae explosions
- Field mostly in the galactic plane



Radio galaxies --IGM

- Typical size: 30 Kpc wide, 300 Kpc long
- Turbulence in central object driven by gravitational/rotational energy of SMBH
- Evidence for expulsion of magnetic helices in lobes





A bit of history...

1908- Hale: Sunspots have strong magnetic fields

1919- Larmor: Dynamo action is introduced

1934- Cowling: Impossibility of axisymmetric dynamo action

- Need for three-dimensional motions

1955- Parker: Cyclonic events and the Γ -effect

1964- Braginskii: Nearly-axisymmetric dynamos

1966- Steebeck, Krause & Rädler: Mean field electrodynamics

- α - ω and α^2 dynamo models

1972- Vainshtein & Zel'dovich: Fast and slow dynamos

1979- Moffatt: Magnetic field generation in electrically conducting fluids

1995- Gilbert & Childress: Stretch, Twist, Fold

Now- Computer models, dynamical systems, spectral theory, cycle expansions, etc...



Mathematical description of a dynamo

Evolution described by the induction equation and Navier-Stokes equation

$$\partial_t \mathbf{B} + \mathbf{U} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{U} + R_m^{-1} \nabla^2 \mathbf{B}$$

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla p + \mathbf{J} \times \mathbf{B} + R^{-1} \nabla^2 \mathbf{U} + \mathbf{F}$$

$$\nabla \cdot \mathbf{B} = \nabla \cdot \mathbf{U} = 0, \quad \mathbf{J} = \nabla \times \mathbf{B}$$

$$R = U l / \nu, \quad R_m = U l / \eta$$

In most astrophysical situations it is assumed that initially

$$|\mathbf{B}| \ll |\mathbf{U}|$$

cf dynamo action in laboratory plasma devices



Some terminology

- **Kinematic**: If initial field is weak, Lorentz force is negligible. Velocity is independent of B . Solve induction for prescribed velocity (eigenvalue problem for growth rate).
- **Fast/Slow**: Kinematic growth rate as $R_m \rightarrow \infty$.
 - *Fast*: Remains positive
 - *Slow*: Negative or approaching zero
- **Large/small**: Characteristic scale of generated field
 - *Small-scale*: Comparable or smaller than the velocity correlation length
 - *Large-scale*: Larger than velocity correlation length
 - Require lack of reflectional symmetry (helicity/rotation)
 - Existence of inverse cascades
 - Mean field effects
 - Mean induction α -effect
 - Turbulent diffusion -effect



Fast dynamo action

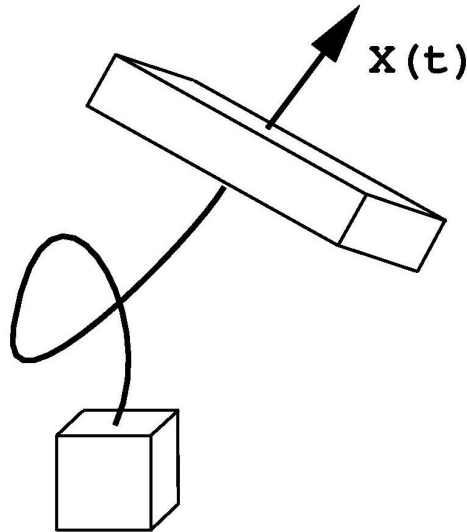
Magnetic field grows if on average rate of field generation exceeds rate of field destruction

- Magnetic field generation due to line stretching by fluid motions
- Magnetic field destruction due to enhanced diffusion

Dynamo growth rate depends on competition between these two effects. In a chaotic flow both effects proceed at an exponential rate.



Chaotic flows



Fluid trajectory given by

$$\dot{\mathbf{X}}(t) = \mathbf{U}(\mathbf{X}(t), t), \quad \mathbf{X}(0) = \mathbf{a}$$

Follow deformation of cube of fluid of initial size δx over a short time δt . New size

$$\delta x (e^{\lambda_1 \delta t}, e^{\lambda_2 \delta t}, e^{\lambda_3 \delta t})$$

If deformation proceeds at exponential rate *on average*, flow is chaotic.

$\lambda_1, \lambda_2, \lambda_3$ are the Lyapunov exponents. ($\lambda_1 + \lambda_2 + \lambda_3 = 0$)

λ_1 Rate of stretching

λ_3 Rate of squeezing

Example of 2-D chaotic flow

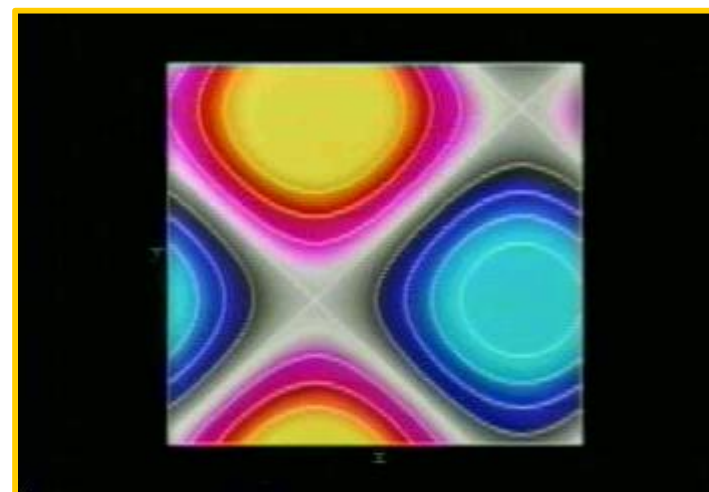
Simple example of smooth solenoidal flow with chaotic streamlines.

$$\mathbf{u}_p = (\psi_z, 0, -\psi_x)$$

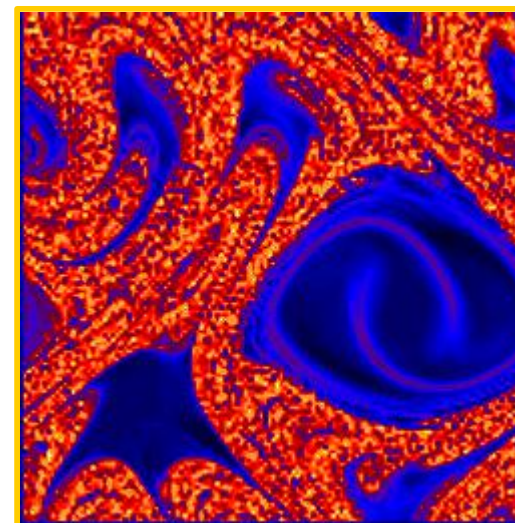
$$\psi = \sqrt{3/2} (\sin(x + \cos t) + \cos(z + \sin t))$$

Red and yellow correspond to trajectories with positive (finite time) Lyapunov exponents $\lambda_1(\mathbf{x}, t)$

λ_1 Rate of divergence. Local stretching
 λ_3 Rate of convergence. Local shrinking



Streamlines



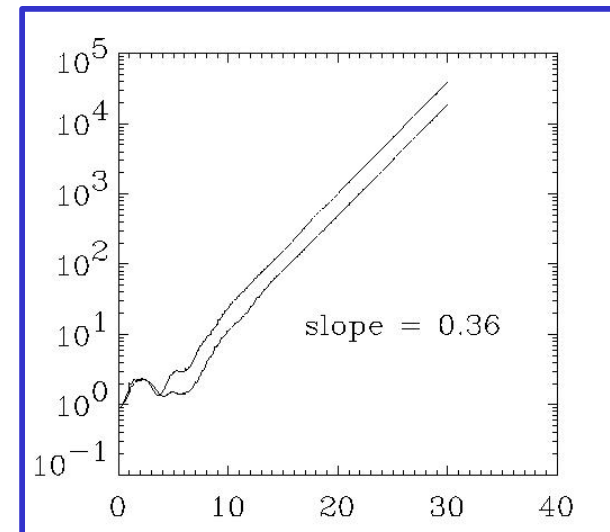
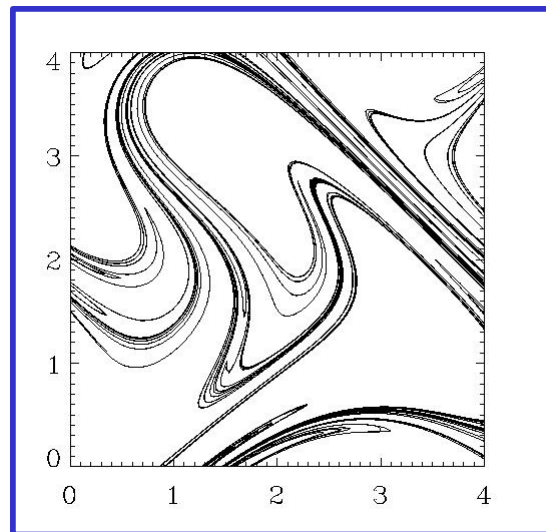
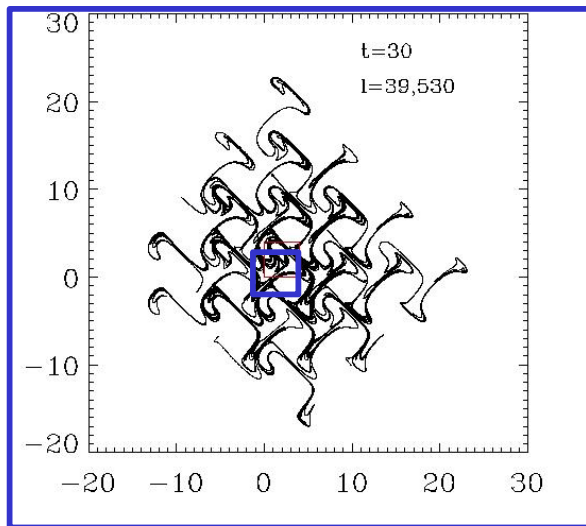
Finite time Lyapunov exponents

Line stretching in a chaotic flow

In a chaotic flow the length of *lines* increases exponentially (on average)

$$l(t) = l_0 \exp(\lambda_T t)$$

λ_T is the *topological entropy*. It satisfies $\lambda_T \geq \lambda_1$





What about dissipation?

Do dynamos operate at the rate ?

$|\kappa_3|$ local rate of growth of *gradients* \Rightarrow dissipation also increases exponentially.

In two dimensions $\kappa_1 = -\kappa_3$. Magnetic field is destroyed as rapidly as it is generated.

$$\mathbf{B} = \nabla \times (A \mathbf{e}_y)$$

$$\left(\partial_t - R_m^{-1} \nabla^2 \right) A + \mathbf{u} \cdot \nabla A = 0$$

In 2D magnetic flux behaves like a *scalar*. Thus

$$\partial_t \langle A^2 \rangle = -2R_m^{-1} \langle |\nabla A|^2 \rangle$$

Two-dimensional dynamo action is *impossible* (Zel'dovich 1957)

Introduce third dimension

Simple modification leads to dynamo action (Galloway & Proctor 1992)

$$\mathbf{u}_0 = (\psi_z, \psi, -\psi_x)$$

Three-dimensional but still y-independent.

$$\mathbf{B}(\mathbf{x}, t) = \mathbf{b}(x, z, t) \exp(\sigma t +iky)$$

For $R_m \gg 1$ and $k = 0.57$
 $\Re(\sigma) \approx 0.3$

However still have $\gamma_1 = -\gamma_3$



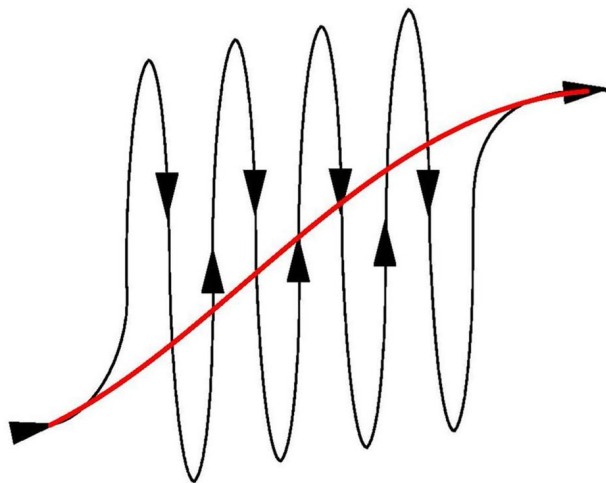


Enhanced diffusion

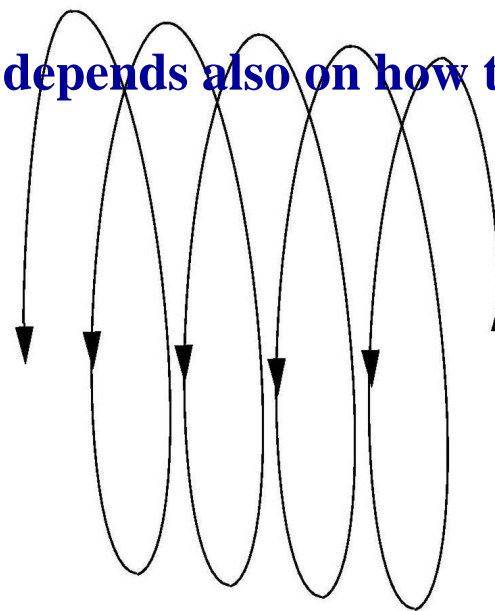
Diffusion of magnetic field is determined by two processes:

- Growth in the magnitude of the gradients of B
- Geometry of the sign reversals of the field lines

Effectiveness of diffusion of a vector field depends also on how the field lines are arranged



Effective



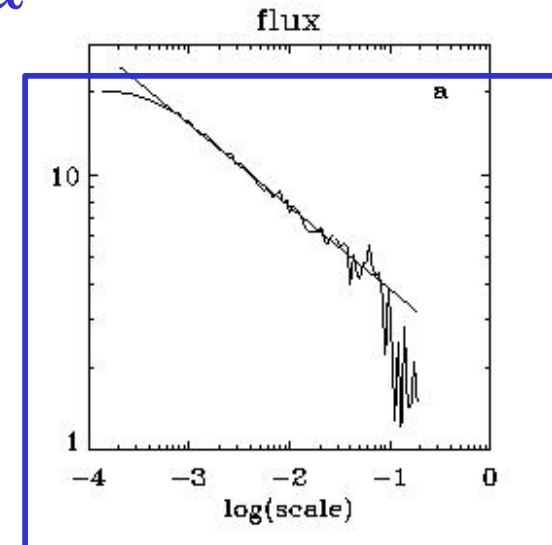
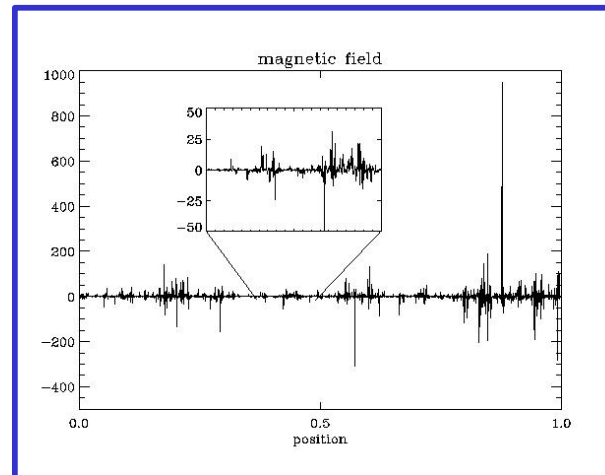
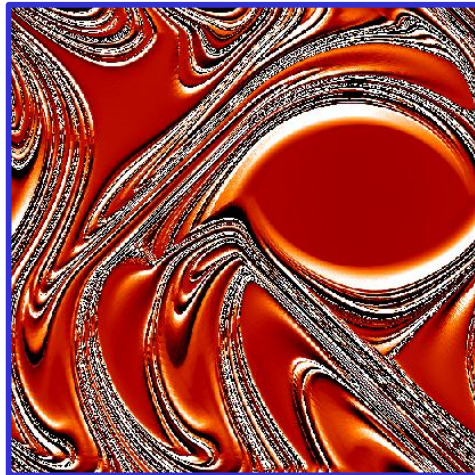
Ineffective

Enhanced diffusion

Effective diffusion of vector quantities depends on magnitude of gradients and *orientation*. Packing becomes important.

$$\mu_s(\varepsilon) = \left| \int_{\varepsilon} B dx \right|$$

$$\chi(\varepsilon) = \frac{\sum_i \mu_s(\varepsilon(i))}{\mu} \approx \varepsilon^{-k}$$



from Cattaneo

k is the cancellation exponent. Measures the singular nature of sign reversals (Du & Ott 1994)



Fast dynamo growth rate

Enhanced diffusion depends on the (exponential) growth of gradients and on field alignment

$$L_1 = \exp(\lambda_1 \delta t)$$

Local stretching

$$L_2 = \exp(-|\lambda_2| \delta t)$$

Local contraction

Conjecture by Du & Ott (1995) for foliated fields as $Rm \rightarrow \infty$

$$\sigma_R = \lim_{\delta t \rightarrow \infty} \frac{1}{\delta t} \langle L_1 L_2^k \rangle$$



Fast dynamo: problems

- **Enhanced diffusion (cancellation exponent) requires global knowledge of geometry of trajectories (very hard).**
- **What is the generalization of the Ott-Du formula?**
 - Give growth rate as a function of velocity statistics.
- **Resulting magnetic field is a (multi)-fractal object. Describe in terms of multi-fractal measures D_q , say.**
 - Give D_q 's as functions of velocity statistics.
- **What happens after kinematic regime?**
 - Magnetic field re-laminarises?
 - How does non-linear saturation occurs?



Saturation in mean field dynamos

Two-scale approach. Separate variables into large and small-scale components. Consider the evolution of the large-scale field

$$\partial_t \langle \mathbf{B} \rangle = \nabla \times \langle \mathbf{u} \times \mathbf{b} \rangle + Rm^{-1} \nabla^2 \langle \mathbf{B} \rangle$$

In kinematic regime linearity of induction equation establishes a linear relationship between mean field and mean emf

$$\langle \mathbf{u} \times \mathbf{b} \rangle_i = \alpha_{ij} \langle B_j \rangle - \epsilon_{ijk} \partial_j \langle B_k \rangle + \dots$$

Isotropic case

α mean induction

β eddy diffusion



Transport coefficients

- In kinematic regime α and β should be determined solely by Rm and the statistics of \mathbf{u}
- α as a pseudo-scalar (tensor) \hat{e} requires lack of reflectional symmetry
- Simple solutions of dynamo equation

$$\mathbf{B} = \mathbf{B}_0(\mathbf{x}) \exp(\sigma t), \quad \nabla \times \mathbf{B}_0 = k \mathbf{B}_0, \quad \sigma = k(\alpha - k)$$

with

$$k_c = \alpha /$$

In large Rm situation α and β should have turbulent values. i.e. independent of Rm

$$\alpha \approx u, \quad \beta \approx u l, \quad k_c l = O(1)$$

Dynamo sets in at small, rather than large scales (bit of a problem)



Saturation in large scale dynamos

- In systems lacking reflectional symmetry mean induction effect (α -effect) leads to growth of large-scale field
- Observations show that in many astrophysical systems *large-scale* field is in equipartition with velocity
- This could easily be achieved if mean induction effect saturates when

$$\langle \mathbf{B} \rangle \approx u$$

- Suggesting a phenomenological non-linear behaviour of the mean induction term of the type

$$\alpha \approx \frac{u}{1 + \langle \mathbf{B} \rangle^2 / u^2}$$

- However there appear to be problems when $Rm \gg 1$.



Nonlinear effects: 2D diffusion

In 2D induction equation becomes scalar transport equation

$$\mathbf{B} = \nabla \times (A \mathbf{e}_y), \quad \left(\partial_t - Rm^{-1} \nabla^2 \right) A + \mathbf{u} \cdot \nabla A = 0$$

With suitable boundary conditions we have

$$\partial_t \langle A^2 \rangle = -2Rm^{-1} \langle |\nabla A|^2 \rangle = -2Rm^{-1} \langle \mathbf{B}^2 \rangle$$

- In order to maintain “turbulent” behaviour as $Rm \rightarrow \infty$ gradients of A must diverge
- Generation of small scale fluctuations increases magnetic field energy

$$\langle |\mathbf{B}|^2 \rangle = \langle \mathbf{B} \rangle^2 Rm$$

- Reasonable energetic constraint $\langle \mathbf{B}^2 \rangle = \mathbf{u}^2$, gives estimate

$$= \frac{u^2}{1 + Rm B^2 / u^2}, \quad B = \langle \mathbf{B} \rangle$$



Effect on velocity field

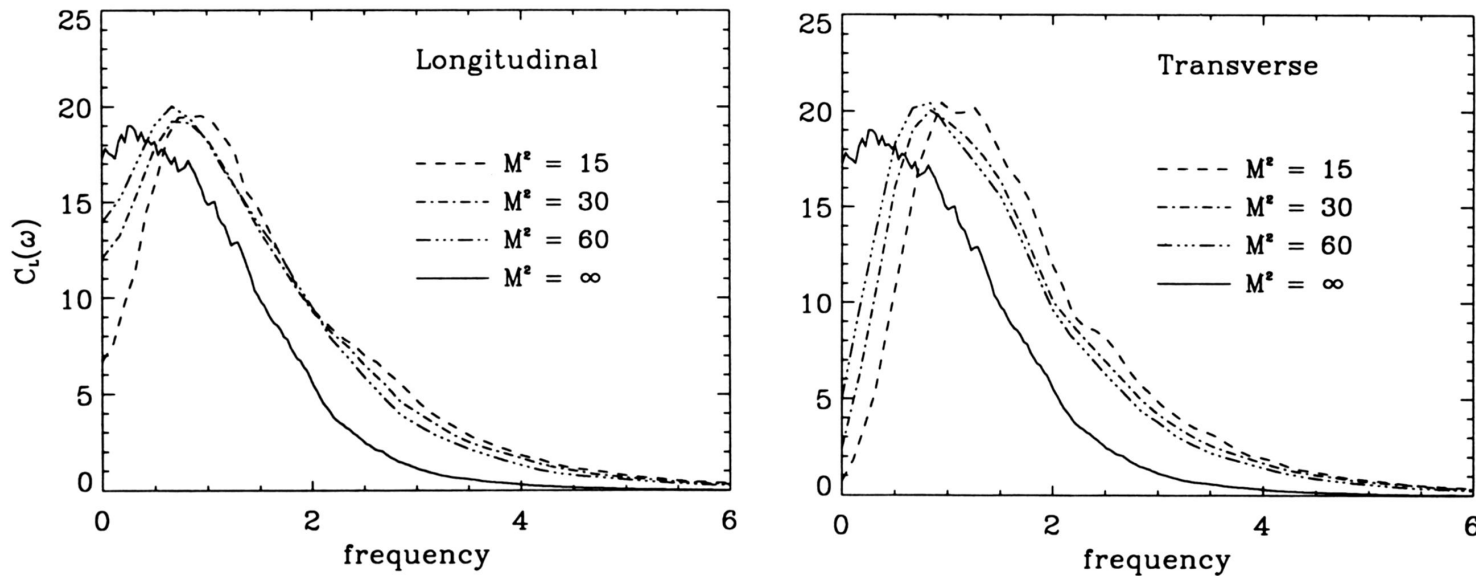
- Assume diffusive behaviour of large scale component of A ($D=$)

$$\partial_t \langle A \rangle = D \nabla^2 \langle A \rangle$$

- With diffusivity given by (Taylor 1921)

$$D = \frac{1}{4} \frac{d}{dt} \langle \xi^2 \rangle \approx \frac{1}{4} C_L(0)$$

$$C_L(\omega) = \int_{-\infty}^{\infty} \langle \mathbf{v}(\mathbf{a}, t) \mathbf{v}(\mathbf{a}, t + s) \rangle \exp i \omega s \, ds$$



from Cattaneo

Turbulence develops a memory



Back to α

- Equivalent expression for α (Moffatt 1964)

$$\alpha = \frac{1}{6} \frac{d}{dt} \langle \mathbf{v} \cdot \nabla_L \times \mathbf{v} \rangle$$

- Kinematically some problems with convergence of integrals as $Rm \rightarrow \infty$
- Non-linearly is there an analogy between 2-D diffusion and 3-D α -effect?
 - Phenomenological argument (Vainshtein & Cattaneo 1991)
 - Closure argument (Kulsrud & Anderson 1992)
 - Quasi-linear closure (Diamond & Gruzinov 1994)



Nonlinear effects

Most nonlinear treatments rely on two statements

- Geometrical (to maintain turbulent rate as $Rm \rightarrow \infty$ something must diverge)
- Dynamical (divergence is energetically impossible)

Assume suitable boundaries

$$\frac{d}{dt} \langle \mathbf{A} \cdot \mathbf{B} \rangle = -2R_m^{-1} \langle \mathbf{B} \cdot \mathbf{J} \rangle \quad (\text{exact})$$

Stationary, uniform mean field

$$\alpha \langle \mathbf{B} \rangle^2 = -R_m^{-1} \langle \mathbf{b} \cdot \mathbf{j} \rangle \quad (\text{exact})$$

From EDQNM, say (Pouquet, Frish & Leorat 1975), get dynamical relationship

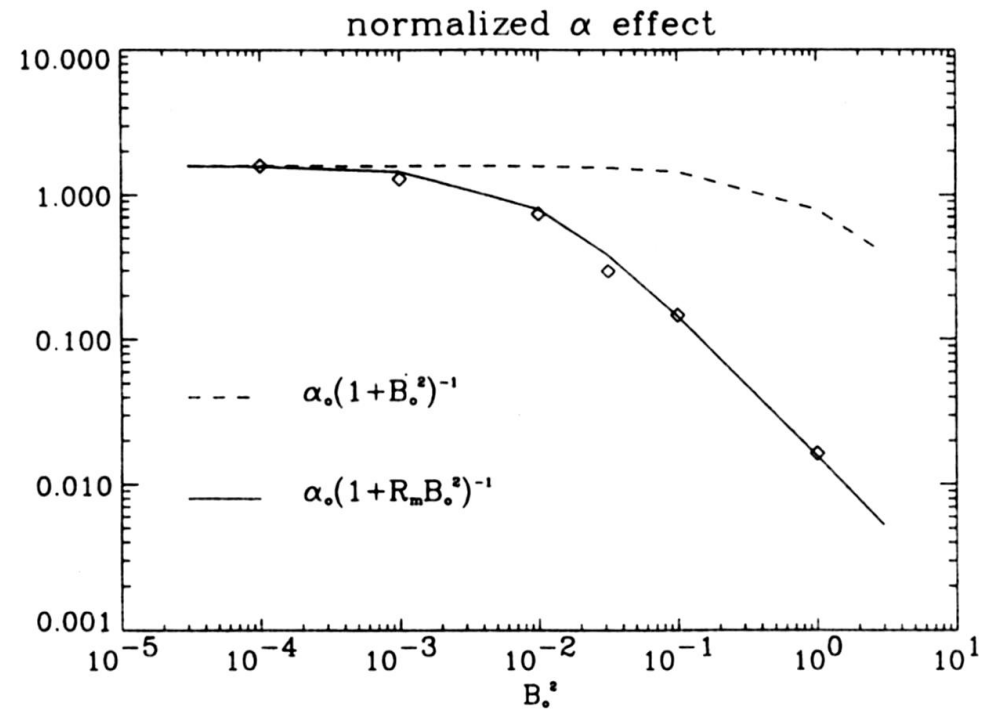
$$\alpha = - \langle \mathbf{u} \cdot ? - \mathbf{b} \cdot \mathbf{j} \rangle \quad (\text{extremely not-exact})$$



Nonlinear effects

Combine to get saturation effects (as before)

$$\alpha = \frac{u}{1 + RmB^2 / u^2}, \quad B = \langle B \rangle$$



from Cattaneo & Hughes



Non-linear saturation: problems

- **Final result is correct but irrelevant**
 - time dependence is neglected
 - large scale gradients are neglected
 - special boundary conditions (no flux of helicity) are assumed
- **Derivation is wrong/suspect**
 - Assumptions about correlation time need justification
 - intermittency effects are neglected (possibly strong in 3D)
 - last expression only valid for (moderate) Rm
- **α -effect is not saturated in laboratory plasmas (rfp)**
- **If everything is correct, how do large-scale equipartition fields get generated?**



Summary/Conclusion

- **Dynamo theory does provide possible framework to explain origin of magnetic fields in widely different astrophysical objects.**

- **At the moment**

- **Large Rm limit**

- **Field development**

- **Field development**

- **Transition from**

- **Does interplay**

- **What is typical**

- **regime?**

- **Is the α -effect suppressed in realistic situations? What about the β -effects?**

- **Does the turbulence develop a long-term memory in 3-D?**

- **Etc. etc. etc.**

Dynamos are sneakier than you think.

dynamical?

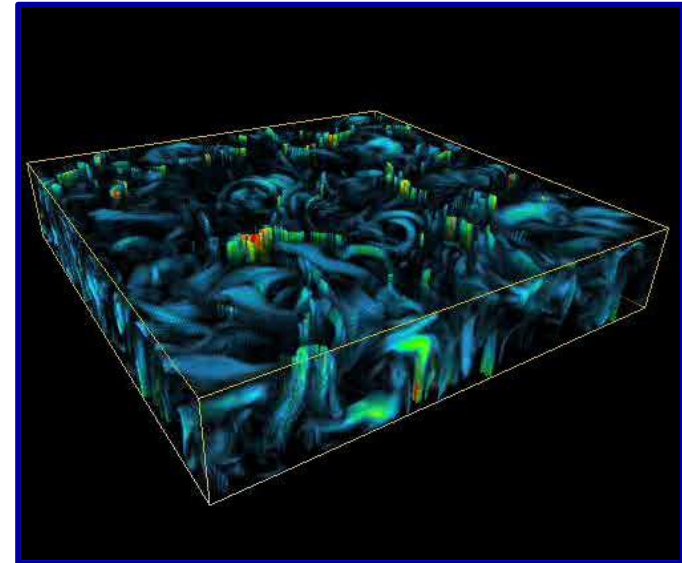
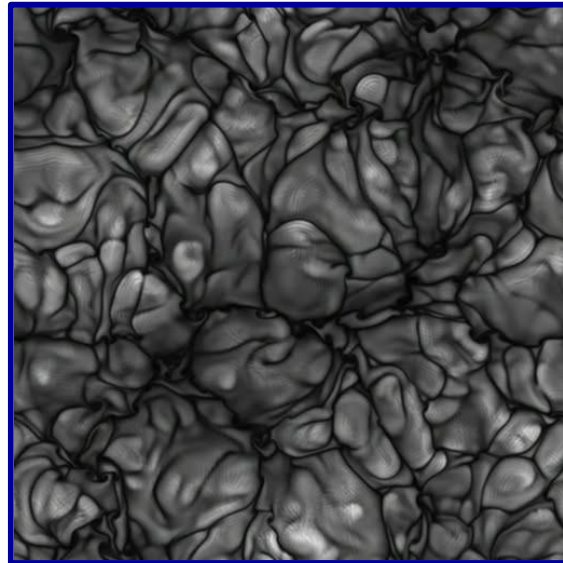
saturated



The end

Some (computer) examples

**Non-rotating
convectively
driven
dynamo**



**Quasi-
geostrophic
driven
dynamo**

