Astrophysical Dynamo Action

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Preliminary considerations

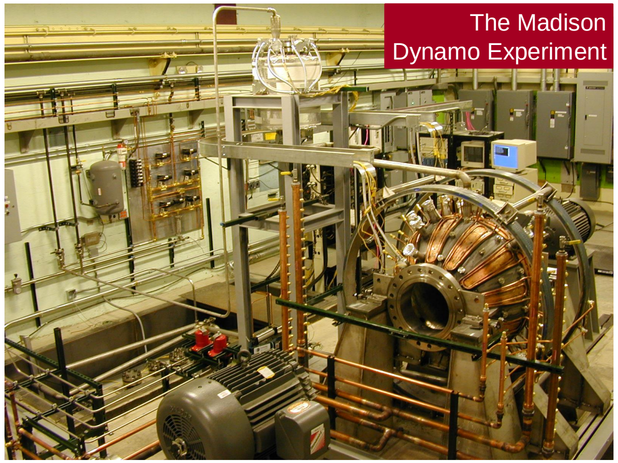
A hydromagnetic dynamo is a sustained mechanism to convert kinetic energy into magnetic energy within the bulk of an electrically conducting fluid

- **Commonly invoked to explain the origin of cosmic magnetic fields**
- **Universe is a strongly magnetized place (now)**
- **WMAP. Universe was not strongly magnetized at recombination**
- **Something must have generated magnetic fields between then and now**

Assume magnetic fields are generated by dynamos

Some examples

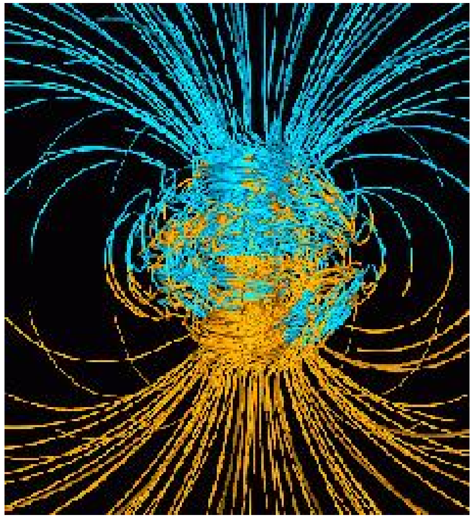
Liquid metal experiments



- Typical size 0.5 – 2.0 m
- Turbulence driven by propellers

Geodynamo

- Size: 6,400 Km
- Turbulence driven by compositional convection in the liquid core
- Evidence for magnetic reversals

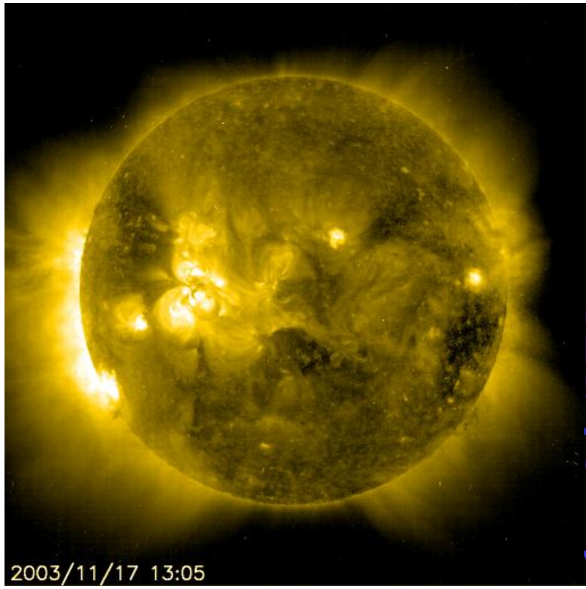


from Glatzmaier

Some examples

Sun (late-type stars)

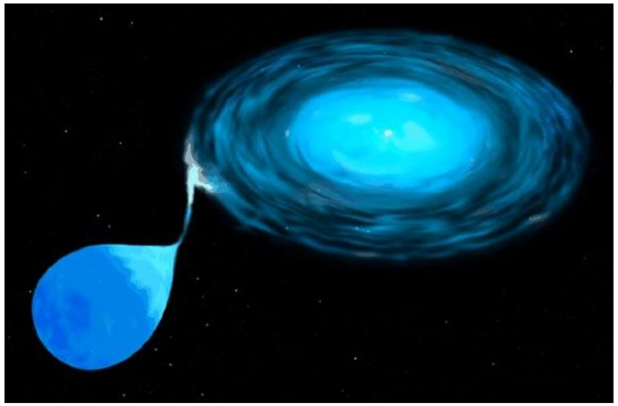
- Size 600,000 Km
- Turbulence driven by thermal convection
- Evidence for activity cycles



from Soho\MDI

Accretion disks

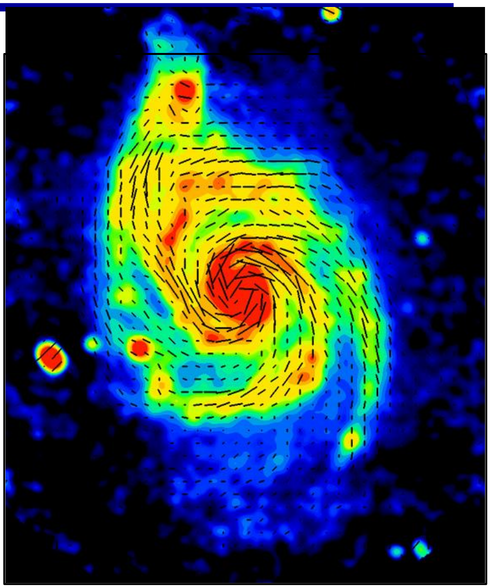
- Typical size varies
- Turbulence driven by MRI



Some examples

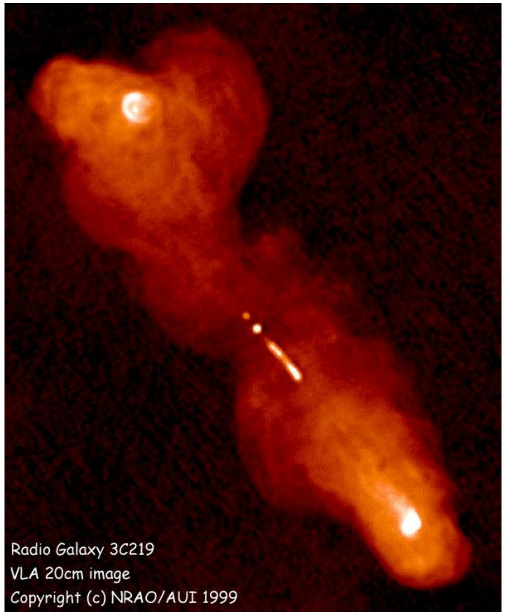
Galaxy

- Typical size: 10^{20} m
- Turbulence driven by supernovae explosions
- Field mostly in the galactic plane



Radio galaxies --IGM

- Typical size: 30 Kpc wide, 300 Kpc long
- Turbulence in central object driven by gravitational/rotational energy of SMBH
- Evidence for expulsion of magnetic helices in lobes



A bit of history…

1908- Hale: Sunspots have strong magnetic fields

1919- Larmor: Dynamo action is introduced

1934- Cowling: Impossibility of axisymmetric dynamo action

- **Need for three-dimensional motions**

1955- Parker: Cyclonic events and the Γ-effect

1964- Braginskii: Nearly-axisymmetric dynamos

1966- Steebeck, Krause & Rädler: Mean field eletrodynamics

- **α-ω and α^2 dynamo models**

1972- Vainshtein & Zel'dovich: Fast and slow dynamos

1979- Moffatt: Magnetic field generation in electrically conducting fluids

1995- Gilbert & Childress: Stretch, Twist, Fold

Now- Computer models, dynamical systems, spectral theory, cycle expansions, etc...

Mathematical description of a dynamo

Evolution described by the *induction equation* and *Navier-Stokes equation*

$$\partial_t \mathbf{B} + \mathbf{U} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{U} + R_m^{-1} \nabla^2 \mathbf{B}$$

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla p + \mathbf{J} \times \mathbf{B} + R^{-1} \nabla^2 \mathbf{U} + \mathbf{F}$$

$$\nabla \cdot \mathbf{B} = \nabla \cdot \mathbf{U} = 0, \quad \mathbf{J} = \nabla \times \mathbf{B}$$

$$\boxed{R = Ul/ \quad , \quad R_m = Ul/\eta}$$

In most astrophysical situations it is assumed that initially

$$|\mathbf{B}| << |\mathbf{U}|$$

cf dynamo action in laboratory plasma devices

Some terminology

- Kinematic: If initial field is weak, Lorentz force is negligible. Velocity is independent of B. Solve induction for prescribed velocity (eigenvalue problem for growth rate).
- Fast/Slow: Kinematic growth rowth rate as $R_m \to \infty$.
  - *Fast:* Remains positive
  - *Slow:* Negative or approaching zero
- Large/small: Characteristic scale of generated field
  - *Small-scale*: Comparable or smaller that the velocity correlation length
  - *Large-scale*: Larger than velocity correlation length
    - Require lack or reflectional symmetry (helicity/rotation)
    - Existence of inverse cascades
    - Mean field effects
      - Mean induction α-effect
      - Turbulent diffusion -effect

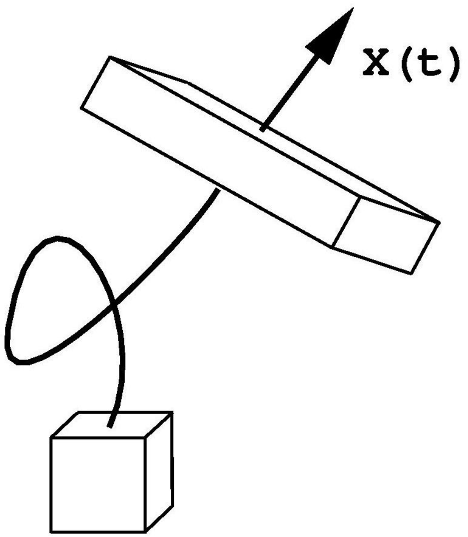
Fast dynamo action

Magnetic field grows if on average rate of *field generation* exceeds rate of *field destruction*

- **Magnetic field generation due to line stretching by fluid motions**
- **Magnetic field destruction due to enhanced diffusion**

Dynamo *growth rate* depends on competition between these two effects. In a *chaotic flow* both effects proceed at an *exponential* rate.

Chaotic flows



Fluid trajectory given by

$$\dot{\mathbf{X}}(t) = \mathbf{U}(\mathbf{X}(t), t), \quad \mathbf{X}(0) = \mathbf{a}$$

Follow deformation of cube of fluid of initial size δx over a short time δt. New size

$$\delta x(e^{\lambda_1 \delta t}, e^{\lambda_2 \delta t}, e^{\lambda_3 \delta t})$$

If deformation proceeds at exponential rate *on average*, flow is *chaotic*.

$\lambda_1, \lambda_2, \lambda_3$ **are the *Lyapunov exponents* .** $(\lambda_1 + \lambda_2 + \lambda_3 = 0)$

λ_1 Rate of stretching λ_3 Rate of squeezing

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Example of 2-D chaotic flow

Simple example of smooth solenoidal flow with chaotic streamlines.

$$\mathbf{u}_p = (\psi_z, 0, -\psi_x)$$

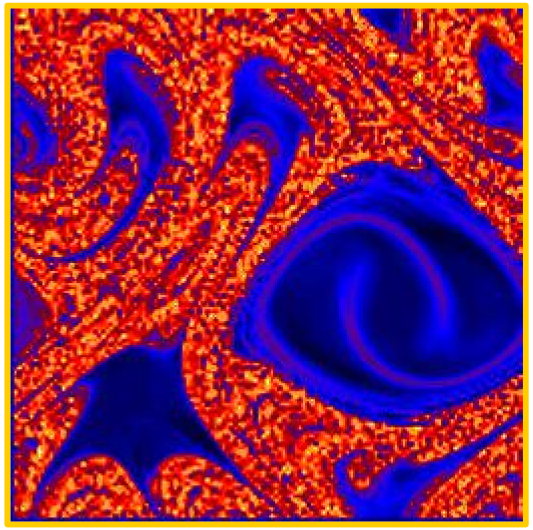
$$\psi = \sqrt{3/2}\,\big(\sin(x + \cos t) + \cos(z + \sin t)\big)$$

Streamlines

Red and yellow correspond to trajectories with positive (finite time) Lyapunov exponents $\lambda_1(\mathbf{x}, t)$

λ_1 Rate of divergence. Local stretching

λ_3 Rate of convergence. Local shrinking



Finite time Lyapunov exponents

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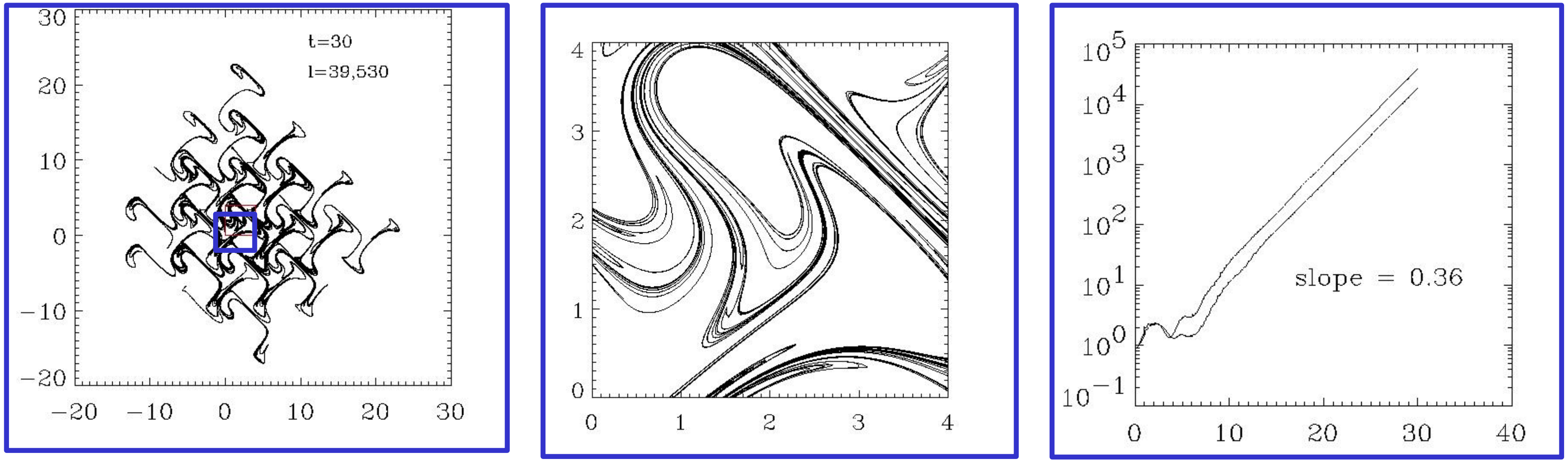
Line stretching in a chaotic flow

In a chaotic flow the length of *lines* increases exponentially (on average)

$$\boxed{l(t) = l_0 \exp(\lambda_T t)}$$

λ_T **is the *topological entropy*. It satisfies** $\lambda_T \geq \lambda_1$



Montana 2004

What about dissipation?

Do dynamos operate at the rate ?ρ?

$|\kappa_3|$ local rate of growth of *gradients* $\Rightarrow$ dissipation also increases exponentially.

In two dimensions $\kappa_1 = -\kappa_3$. Magnetic field is destroyed as rapidly as it is generated.

$$\mathbf{B} = \nabla \times (A\mathbf{e}_y)$$

$$\left(\partial_t - R_m^{-1}\nabla^2\right)A + \mathbf{u}\cdot\nabla A = 0$$

In 2D magnetic flux behaves like a *scalar*. Thus

$$\boxed{\partial_t \langle A^2 \rangle = -2R_m^{-1}\left\langle |\nabla A|^2 \right\rangle}$$

Two-dimensional dynamo action is *impossible* (Zel'dovich 1957)

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Introduce third dimension

Simple modification leads to dynamo action (Galloway & Proctor 1992)

$$\mathbf{u}_0 = (\psi_z, \psi, -\psi_x)$$

Three-dimensional but still y-independent.

$$\mathbf{B}(\mathbf{x},t) = \mathbf{b}(x,z,t)\exp(\sigma t + iky)$$

$$\boxed{\begin{array}{c}\text{For } R_m >> 1 \text{ and } k = 0.57 \\ \Re(\sigma) \approx 0.3\end{array}}$$

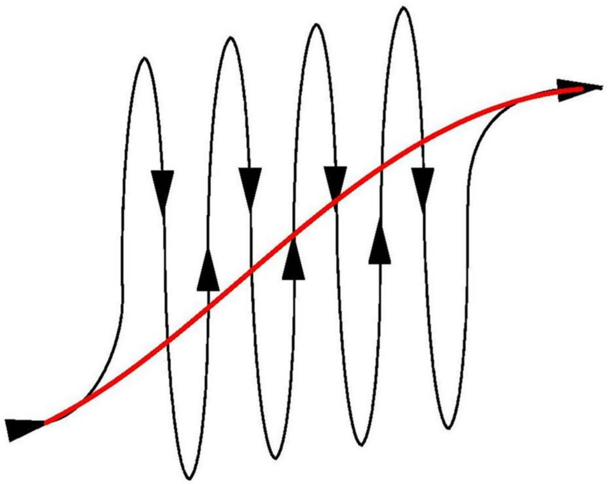
However still have ?$_1$ = - ?$_3$

Enhanced diffusion

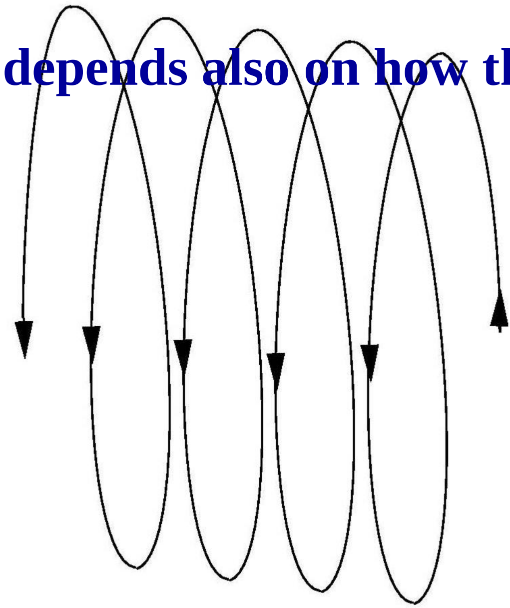
Diffusion of magnetic field is determined by two processes:

- Growth in the magnitude of the ***gradients*** of B
- Geometry of the ***sign reversals*** of the field lines

Effectiveness of diffusion of a *vector field* depends also on how the field lines are arranged



Effective



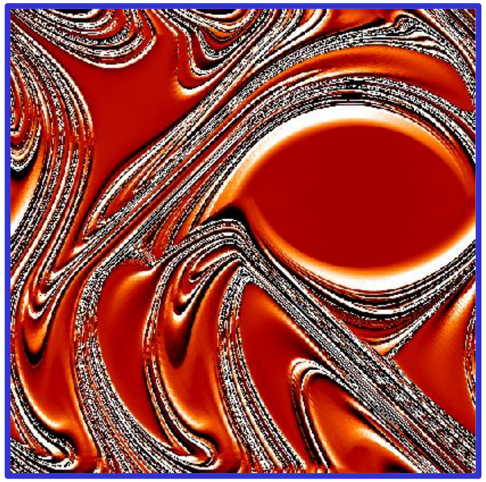
Ineffective

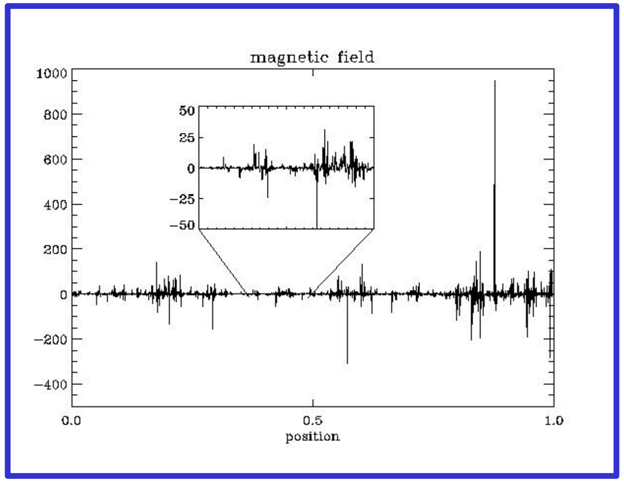
Enhanced diffusion

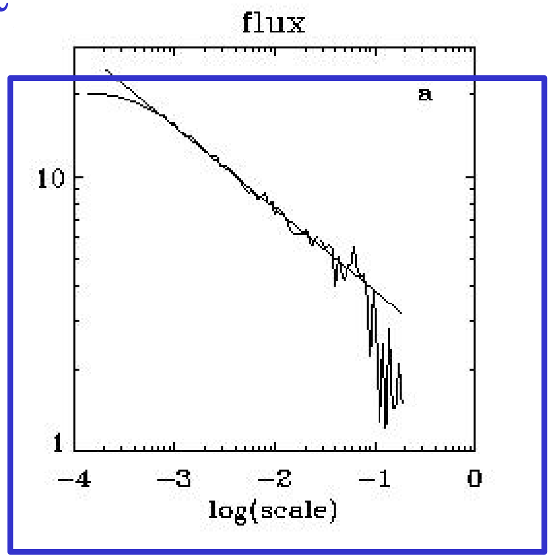
Effective diffusion of vector quantities depends on magnitude of gradients and *orientation*. Packing becomes important.

$$\mu_s(\varepsilon) = \left| \int_\varepsilon B dx \right|$$

$$\chi(\varepsilon) = \frac{\sum_i \mu_s(\varepsilon(i))}{\mu(1)} \approx \varepsilon^{-\kappa}$$







from Cattaneo

k **is the cancellation exponent. Measures the singular nature of sign reversals (Du & Ott 1994)**

Fast dynamo growth rate

Enhanced diffusion depends on the (exponential) growth of gradients and on field alignment

$L_1 = \exp(\lambda_1 \delta t)$	**Local stretching**
$L_2 = \exp(-\lvert\lambda_2\rvert \delta t)$	**Local contraction**

Conjecture by Du & Ott (1995) for foliated fields as *Rm* $\rightarrow \infty$

$$\boxed{\sigma_R = \operatorname*{Lim}_{\delta t \to \infty} \frac{1}{\delta t} \left\langle L_1 L_2^k \right\rangle}$$

Fast dynamo: problems

- Enhanced diffusion (cancellation exponent) requires global knowledge of geometry of trajectories (very hard).
- What is the generalization of the Ott-Du formula?
  - Give growth rate as a function of velocity statistics.
- Resulting magnetic field is a (multi)-fractal object. Describe in terms of multi-fractal measures *Dq*, say.
  - Give *Dq's* as functions of velocity statistics.
- What happens after kinematic regime?
  - Magnetic field re-laminarises?
  - How does non-linear saturation occurs?

Saturation in mean field dynamos

Two-scale approach. **Separate variables into large and small-scale components. Consider the evolution of the large-scale field**

$$\partial_t \langle \mathbf{B} \rangle = \nabla \times \langle \mathbf{u} \times \mathbf{b} \rangle + Rm^{-1} \nabla^2 \langle \mathbf{B} \rangle$$

In kinematic regime linearity of induction equation establishes a linear relationship between mean field and mean emf

$$\langle \mathbf{u} \times \mathbf{b} \rangle_i = \alpha_{ij} \langle B_j \rangle - \quad {}_{ijk} \partial_j \langle B_k \rangle + \ldots$$

Isotropic case

aá mean induction

ßá eddy diffusion

Transport coefficients

- In kinematic regime a¥and ß¥should be determined solely by *Rm* and the statistics of u
- aXs a pseudo-scalar (tensor) è requires lack of reflectional symmetry
- Simple solutions of dynamo equation

$$\mathbf{B} = \mathbf{B_0}(\mathbf{x})\exp(\sigma t), \quad \nabla \times \mathbf{B_0} = k\mathbf{B_0}, \quad \sigma = k(\alpha - k \quad)$$

with

$$k_c = \alpha /$$

In large *Rm* situation aǎnd ßǎshould have turbulent values. i.e. independent of *Rm*

$$\alpha \approx u, \qquad \approx u\text{l}, \quad k_c\text{l} = O(1)$$

Dynamo sets in at small, rather than large scales (bit of a problem)

Saturation in large scale dynamos

- **In systems lacking reflectional symmetry mean induction effect (α-effect) leads to growth of large-scale field**
- **Observations show that in many astrophysical systems *large-scale* field is in equipartition with velocity**
- **This could easily be achieved if mean induction effect saturates when**

$$\left|\langle \mathbf{B} \rangle\right| \approx u$$

- **Suggesting a phenomenological non-linear behaviour of the mean induction term of the type**

$$\alpha \approx \frac{u}{1+\langle \mathbf{B} \rangle^2 / u^2}$$

- **However there appear to be problems when *Rm* >>1.**

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Nonlinear effects: 2D diffusion

In 2D induction equation becomes scalar transport equation

$$\mathbf{B} = \nabla \times (A\mathbf{e}_y), \qquad \left(\partial_t - R_m^{-1}\nabla^2\right)A + \mathbf{u}\cdot\nabla A = 0$$

With suitable boundary conditions we have

$$\partial_t \left\langle A^2 \right\rangle = -2Rm^{-1}\left\langle |\nabla A|^2 \right\rangle = -2Rm^{-1}\left\langle \mathbf{B}^2 \right\rangle$$

- **In order to maintain "turbulent" behaviour as *Rm* $\rightarrow \infty$ gradients of *A* must diverge**
- **Generation of small scale fluctuations increases magnetic field energy**

$$\left\langle |\mathbf{B}|^2 \right\rangle = \left\langle \mathbf{B} \right\rangle^2 Rm$$

- **Reasonable energetic constraint $\langle B^2 \rangle = \rho u^2$, gives estimate**

$$= \frac{ul}{1 + RmB^2/u^2}, \qquad B = \left|\left\langle \mathbf{B} \right\rangle\right|$$

Effect on velocity field

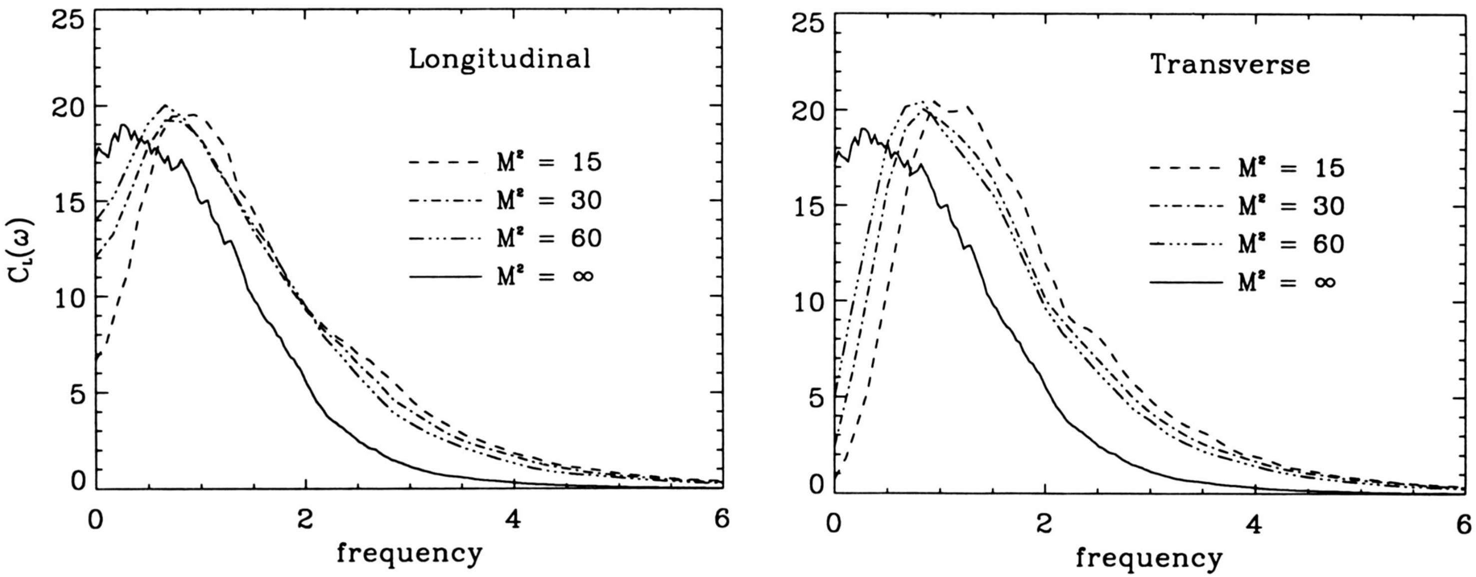
- **Assume diffusive behaviour of large scale component of A (D=)**

$$\partial_t \langle A \rangle = D \nabla^2 \langle A \rangle$$

- **With diffusivity given by (Taylor 1921)**

$$D = \frac{1}{4}\frac{d}{dt}\langle \xi^2 \rangle \approx \frac{1}{4} C_L(0)$$

$$C_L(\omega) = \int_{-\infty}^{\infty} \langle \mathbf{v}(\mathbf{a},t)\mathbf{v}(\mathbf{a},t+s) \rangle \exp i\omega s \; ds$$



from Cattaneo

Turbulence develops a memory

Back to aŸ

- **Equivalent expression for α (Moffatt 1964)**

$$\alpha = \frac{1}{6}\frac{d}{dt}\left\langle ?^{\text{º}}\nabla_L \times ? \right\rangle$$

- **Kinematically some problems with convergence of integrals as *Rm* $\rightarrow \infty$**

- **Non-linearly is there an analogy between 2-D diffusion and 3-D α-effect?**
  - **Phenomenological argument (Vainshtein & Cattaneo 1991)**
  - **Closure argument (Kulsrud & Anderson 1992)**
  - **Quasi-linear closure (Diamond & Gruzinov 1994)**

Nonlinear effects

Most nonlinear treatments rely on two statements

- **Geometrical (to maintain turbulent rate as *Rm* $\to \infty$ something must diverge)**
- **Dynamical (divergence is energetically impossible)**

Assume suitable boundaries

$$\frac{d}{dt}\langle \mathbf{A}\cdot\mathbf{B}\rangle = -2R_m^{-1}\langle \mathbf{B}\cdot\mathbf{J}\rangle \quad \text{(exact)}$$

Stationary, uniform mean field

$$\alpha\langle \mathbf{B}\rangle^2 = -R_m^{-1}\langle \mathbf{b}\cdot\mathbf{j}\rangle \quad \text{(exact)}$$

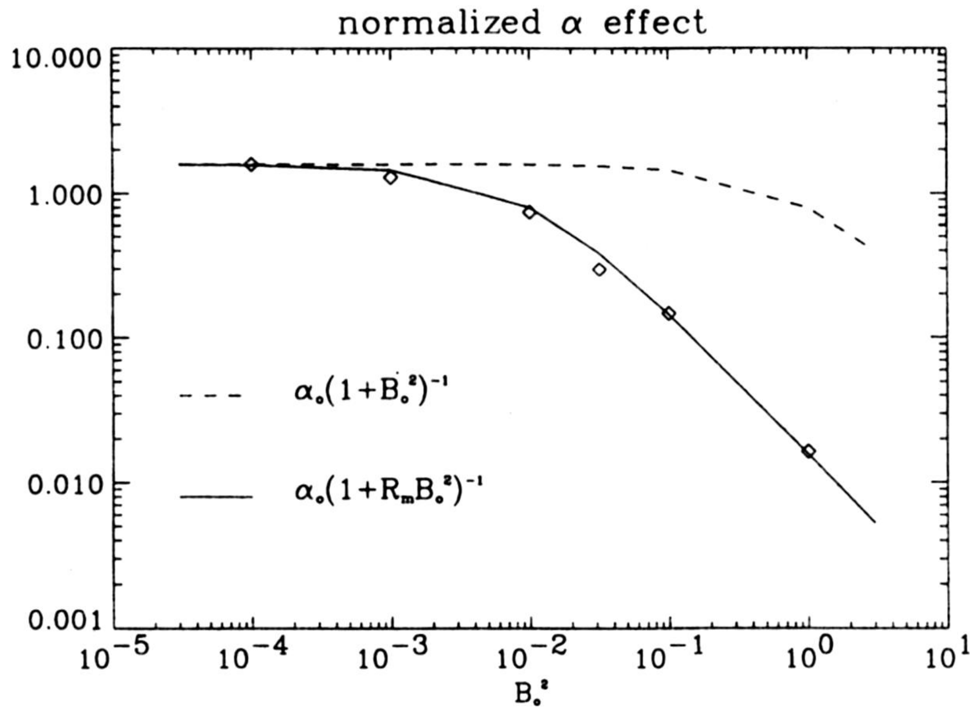
From EDQNM, say (Pouquet, Frish & Leorat 1975), get dynamical relationship

$$\alpha = -\ \langle \mathbf{u}\cdot ?\ -\mathbf{b}\cdot\mathbf{j}\rangle \quad \text{(extremely not-exact)}$$

Nonlinear effects

Combine to get saturation effects (as before)

$$\alpha = \frac{u}{1 + RmB^2/u^2}, \qquad B = \langle \mathbf{B} \rangle$$



from Cattaneo & Hughes

Non-linear saturation: problems

- **Final result is correct but irrelevant**
  - **time dependence is neglected**
  - **large scale gradients are neglected**
  - **special boundary conditions (no flux of helicity) are assumed**

- **Derivation is wrong/suspect**
  - **Assumptions about correlation time need justification**
  - **intermittency effects are neglected (possibly strong in 3D)**
  - **last expression only valid for (moderate) *Rm***

- **α-effect is not saturated in laboratory plasmas (rfp)**

- **If everything is correct, how do large-scale equipartition fields get generated?**

Summary/Conclusion

- Dynamo theory does provide possible framework to explain origin of magnetic fields in widely different astrophysical objects.

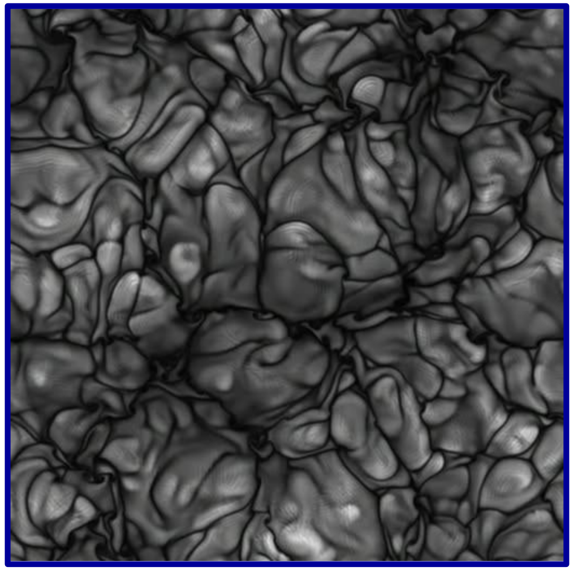
- At the momen
- Large *Rm* lim
  - Field deve
  - Field deve
- Transition fr
  - Does inte dynamical?
  - What is ty turated regime?
  - Is the α-effect suppressed in realistic situations? What about the -effects?
  - Does the turbulence develop a long-term memory in 3-D?
  - Etc. etc. etc.

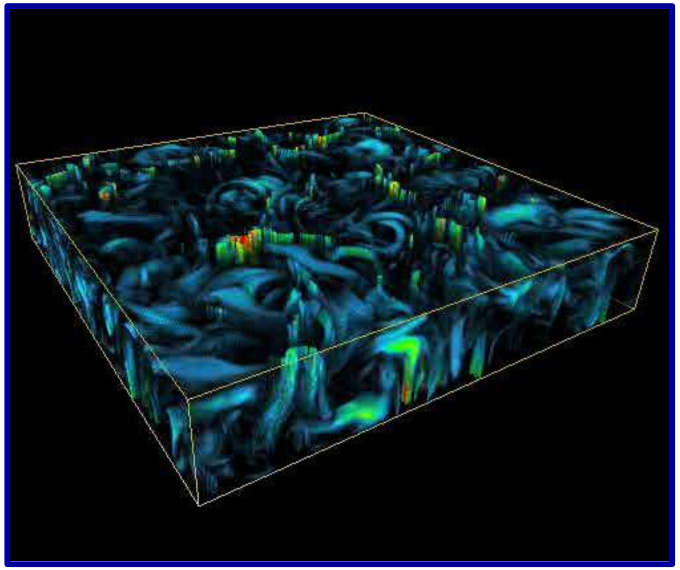
Dynamos are sneakier than you think.

The end

Some (computer) examples

Non-rotating convectively driven dynamo





Quasi-geostrophic driven dynamo