Non-diffusive transport in plasma turbulence: a fractional diffusion approach

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Beyond the standard diffusive transport paradigm

•Recent experimental and theoretical evidence suggests that transport in magnetically confined plasmas deviates from the standard diffusion paradigm:

$$\partial_t P = \partial_x \left[\chi(x,t) \partial_x P \right] + S(x,t)$$

•Examples include:

- •Confinement time scaling of L-mode plasma
- •Fast propagation and non-local transport phenomena
- •Non-Gaussianity and long-range correlations of fluctuations
- •Anomalous diffusion and non-Gaussian pdfs in tracer transport studies

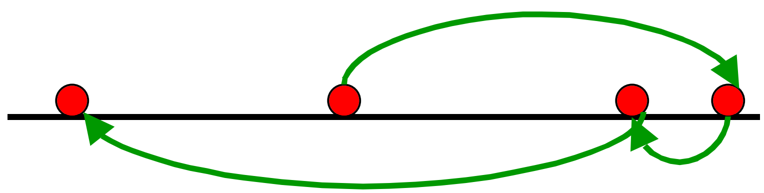
•The standard diffusion paradigm breaks down because it rests on restrictive assumptions including:

- •Space locality(Fick's law)

$$Q = -\chi \partial_x P$$

Flux depends on local quantities

- •Time locality (Markovian assumption)
- •Underlying "microscopic" un-correlated, Gaussian stochastic process (Brownian walk)

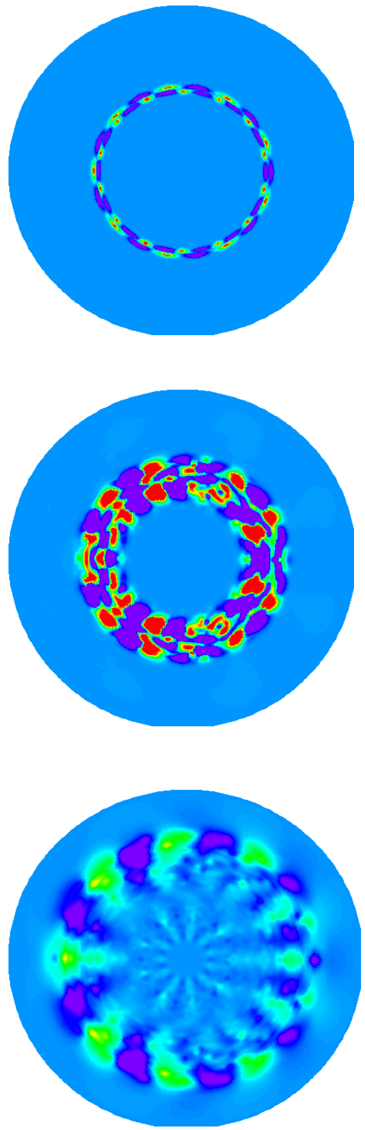


Our objective is to develop transport models that go beyond the restrictive assumptions of the standard diffusion paradigm.

The model is based on the use of fractional derivative operators that incorporate in a unified way non-Fickian transport, "memory effects" (i.e. non-Markovian transport) and non-Gaussian scaling.

Turbulence model

As a concrete case study to motivate and test the model, we consider transport in 3-D pressure-gradient-driven plasma turbulence.



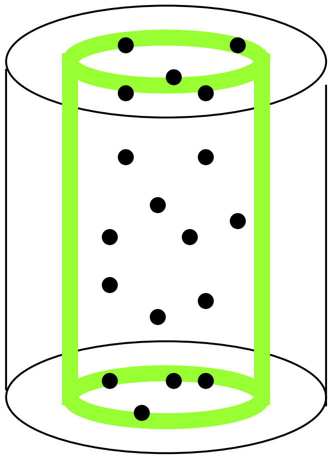
$$\left(\partial_t + \tilde{V}\cdot\nabla\right)\nabla^2_{per}\tilde{\Phi} = \frac{B_0}{m_i n_0 r_c}\frac{1}{r}\frac{\partial\tilde{p}}{\partial\theta} - \frac{1}{\eta m_i n_0 R_0}\nabla^2_{\parallel}\tilde{\Phi} + \mu\nabla^4_{\parallel}\tilde{\Phi}$$

$$\left(\partial_t + \tilde{V}\cdot\nabla\right)\tilde{p} = \frac{\partial\langle p\rangle}{\partial r}\frac{1}{r}\frac{\partial\tilde{\Phi}}{\partial\theta} + \chi_{per}\nabla^2_{per}\tilde{p} + \chi_{\parallel}\nabla^2_{\parallel}\tilde{p}$$

$$\frac{\partial\langle p\rangle}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}r\left\langle\tilde{V}_r\,\tilde{p}\right\rangle = S_0 + D\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\langle p\rangle}{\partial r}\right)$$

Carreras, et.al., Phys.Plasmas 3, 2903 (1996).

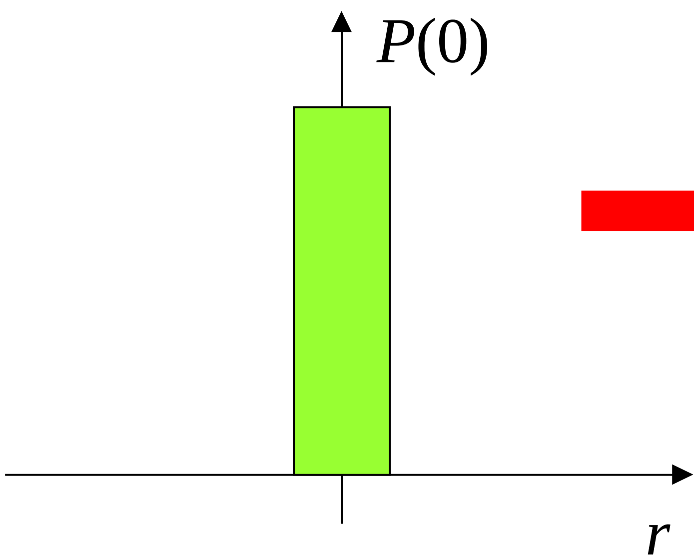
Tracer transport

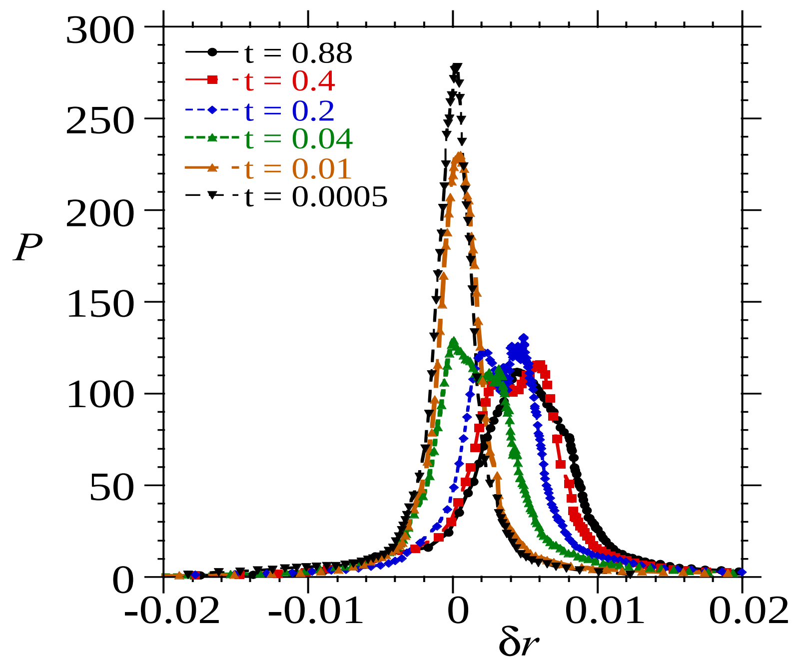


Focus on tracers following the ***E*x*B*** flow velocity

$$\frac{d\vec{r}}{dt} = \frac{1}{B^2}\nabla\tilde{\Phi}\times\vec{B}$$

Initial condition: localized in radius, uniformly distributed in *z* and θ

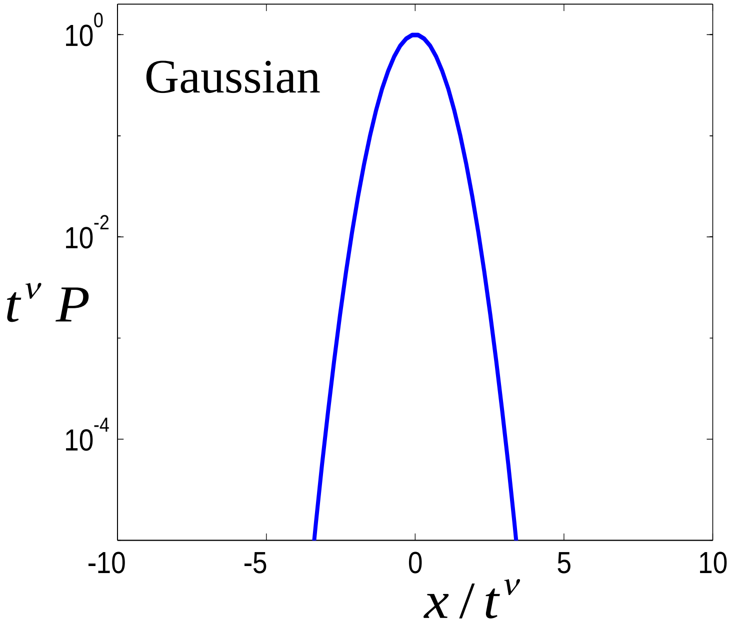


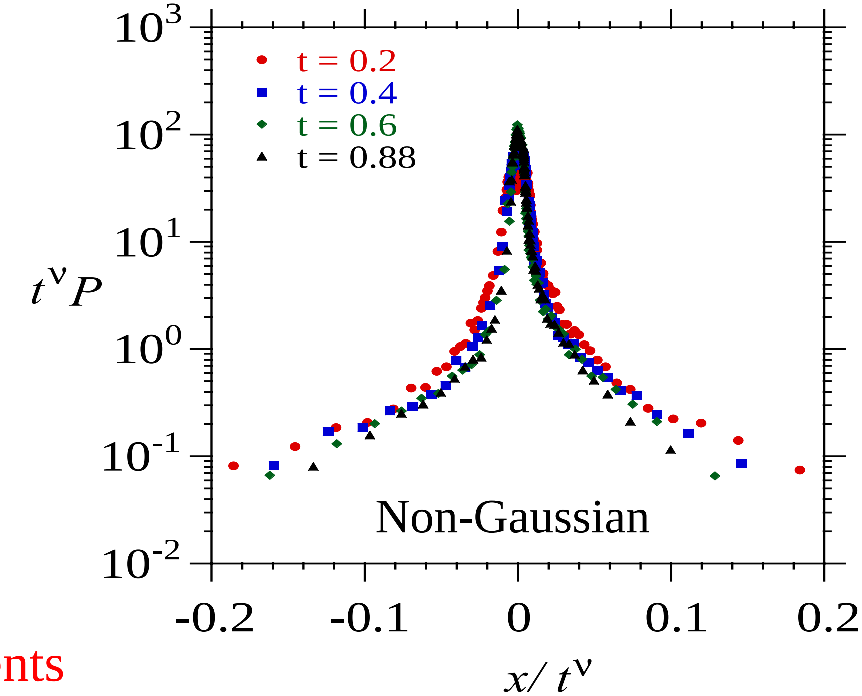


Standard diffusion

Plasma turbulence

Probability density function





Moments

Diffusive scaling $\nu = 1/2$

$$\langle x^n \rangle \sim t^{n\nu}$$

Anomalous scaling super-diffusion $\nu \sim 2/3$

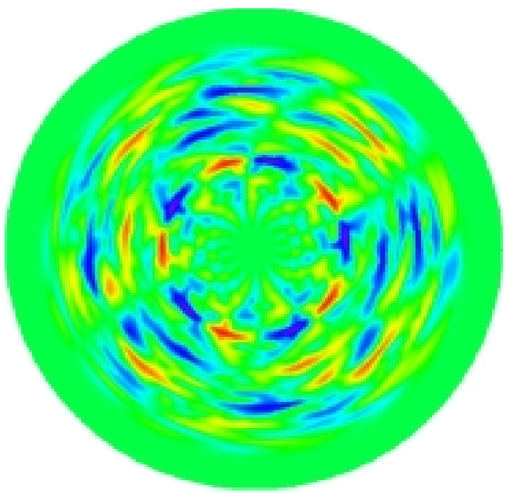
Model

$$\partial_t P = \chi \partial_x^2 P$$

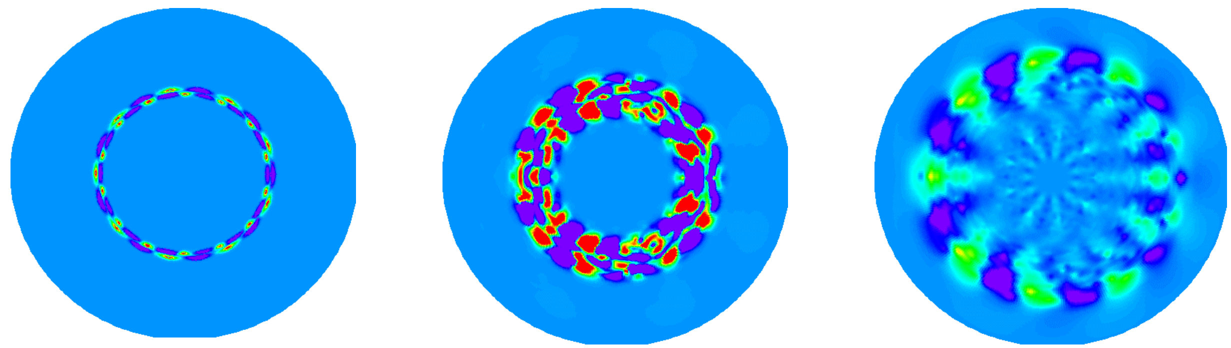
??????????

What causes this non-diffusive transport?

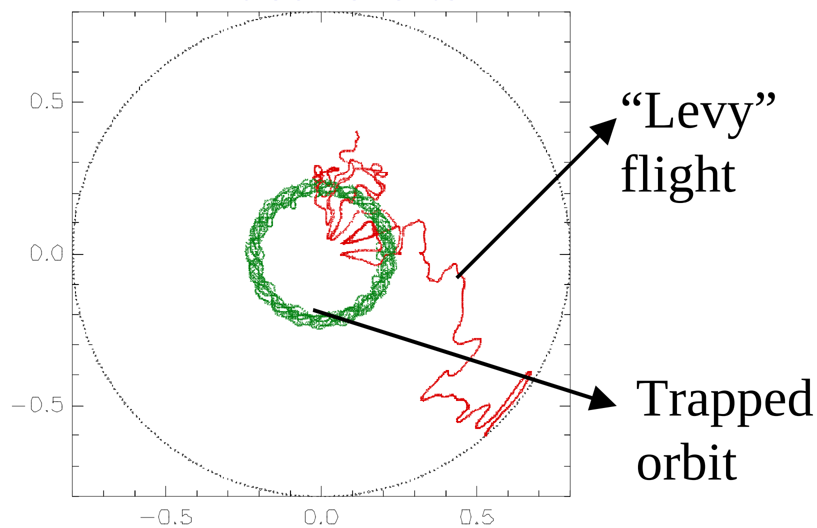
E*x*B flow velocity eddies induce large tracer trapping that leads to temporal non-locality



“Avalanche like” phenomena induce large tracer displacements that lead to spatial non-locality



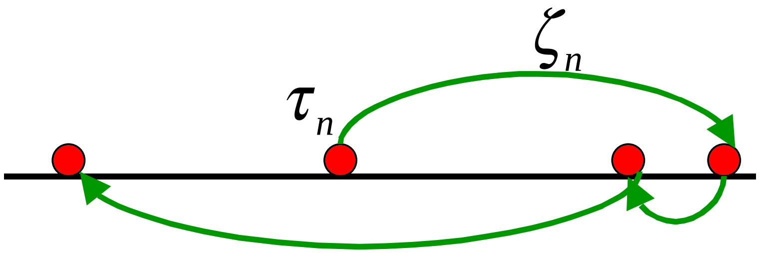
Tracer orbits



The combination of tracer trapping and flights leads to anomalous diffusion

Continuous Time Random Walk Model

(Montroll-Weiss 1965)



τ_n = waiting time $\quad\quad$ $\psi(\tau)$ = waiting time pdf

ζ_n = jump $\quad\quad$ $\lambda(\zeta)$ = jump size pdf

$$P(x,t) = \underbrace{\delta(x)\int_t^{\infty}\psi(t')\,dt'}_{\text{Contribution from particles that have not moved during } (0,t)} + \underbrace{\int_0^t \psi(t-t')\left[\int_{-\infty}^{\infty}\lambda(x-x')\,P(x',t')\,dx'\right]dt'}_{\text{Contribution from particles located at } x' \text{ and jumping to } x \text{ during } (0,t)}$$

Solution in terms of Fourier-Laplace transforms

$$\hat{\tilde{P}}(k,s) = \frac{1-\tilde{\psi}(s)}{s}\,\frac{1}{1-\tilde{\psi}(s)\,\hat{\lambda}(k)}$$

Standard diffusive case

$$\hat{\tilde{P}}(k,s) = \frac{1-\tilde{\psi}(s)}{s}\,\frac{1}{1-\tilde{\psi}(s)\,\hat{\lambda}(k)}$$

$$\left\{ \begin{array}{ll} \psi(\tau) \sim e^{-\mu\tau} & \text{Trapping pdf} \\ \lambda(\zeta) \sim e^{-\zeta^2/2\sigma} & \text{Jumps pdf} \end{array} \right.$$

$1/\mu$ and σ transport scales

Continuum limit $\quad s\,\hat{\tilde{P}} - 1 = -k^2\,\hat{\tilde{P}}$

Laplace transform $\quad L\left[\partial_t \tilde{P}\right] = s\,\tilde{P} - \delta(x)$

Fourier transform $\quad F\left[\partial^2{}_x \hat{P}\right] = -k^2\,\hat{P}$

$$\partial_t P = \chi\,\partial_x^2 P$$

Non-diffusive case

$$\hat{\tilde{P}}(k,s) = \frac{1-\tilde{\psi}(s)}{s} \frac{1}{1-\tilde{\psi}(s)\,\hat{\lambda}(k)}$$

$$\begin{cases} \psi(\tau) \sim \tau^{-(\beta+1)} & \text{Trapping pdf} \\ \lambda(\zeta) \sim |\zeta|^{-(\alpha+1)} & \text{Jumps pdf} \end{cases}$$

No characteristic transport scale $\langle \tau \rangle \to \infty$ $\langle \zeta \rangle \to \infty$

Continuum limit $s^{\beta}\,\hat{\tilde{P}} - s^{\beta-1} = -\chi\,|k|^{\alpha}\,\hat{\tilde{P}}$

Laplace transform $L[??] = s^{\beta}\,\tilde{P} - s^{\beta-1}\,\delta(x)$

Fourier transform $F[??] = -|k|^{\alpha}\,\hat{P}$

$\Rightarrow$???????????

There are no simple differential operators satisfying these relations

Fractional derivatives

Fractional space derivative

$$\partial^{\alpha}_{|x|} P = c\left[{}^{-}\partial^{\alpha}_{x} + {}^{+}\partial^{\alpha}_{x} \right] P$$

$${}^{-}\partial^{\alpha}_{x} P = \frac{1}{\Gamma(m-\alpha)} \partial^{m}_{x} \int_{-\infty}^{x} \frac{P(y)}{(x-y)^{\alpha-m+1}} \, dy$$

$$m-1 \le \alpha < m$$

$$F\left[\partial^{\alpha}_{|x|} \hat{P} \right] = -|k|^{\alpha} \hat{P}$$

Fractional time derivative

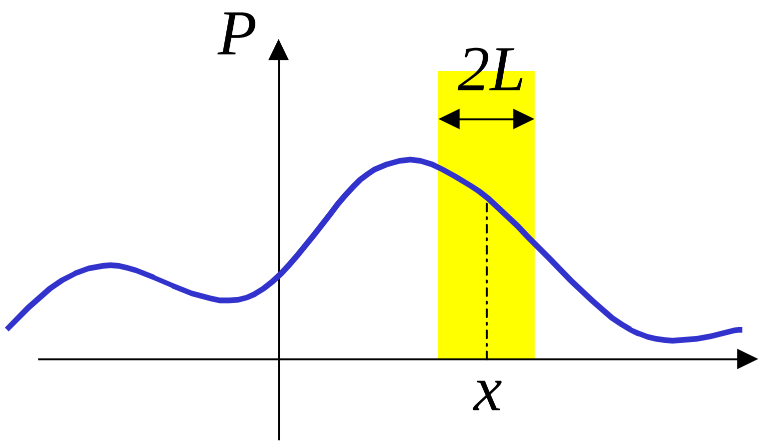
$$\partial^{\beta}_{t} P = \frac{1}{\Gamma(1-\beta)} \int_{0}^{t} \frac{\partial_{\tau} P(\tau)}{(t-\tau)^{\beta}} \, d\tau$$

$$L\left[\partial^{\beta}_{t} \tilde{P} \right] = s^{\beta} \tilde{P} - s^{\beta-1} \delta(x)$$

$$\boxed{\partial^{\beta}_{t} P = \chi \, \partial^{\alpha}_{|x|} P}$$

Fractional transport model

Local and non-local transport



$$\langle P \rangle_L (x) = \frac{1}{2L} \int_L P(x+s)\, ds$$

L="radius of influence"

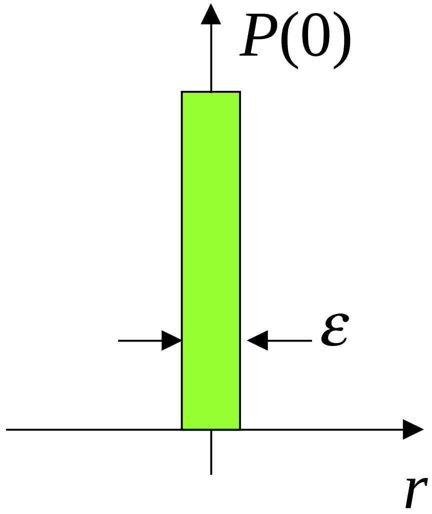
Local
diffusive
transport

$$\partial_t \; P = \chi \partial_x^2 \; P = 6\chi \; \frac{\langle P \rangle_L - P}{L^2} \quad \text{as} \quad L \to 0$$

Non-local
fractional
diffusive
transport

$$\partial_t^\beta \; P = D \partial_{|x|}^\alpha \; P = c\; \partial_x \int_{-\infty}^{x} \frac{P(y)}{(x-y)^\alpha} dy + \mathsf{L}$$

Initial value problem



$$\partial_t^{\beta} P = \chi \, \partial_{|x|}^{\alpha} P$$

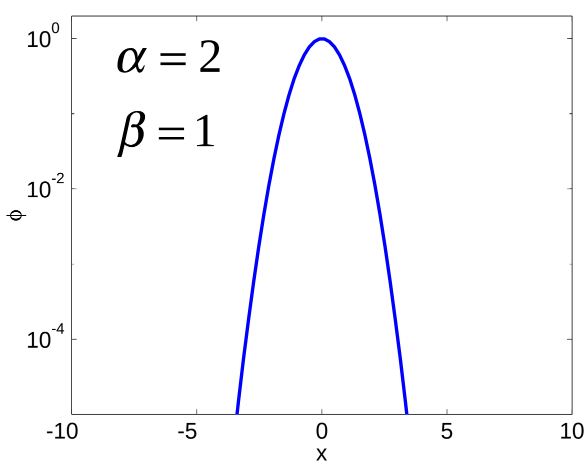
$$\eta = x \, t^{-\beta/\alpha}$$ Self-similar variable

$$\delta = \varepsilon \, t^{-\beta/\alpha}$$ Small parameter

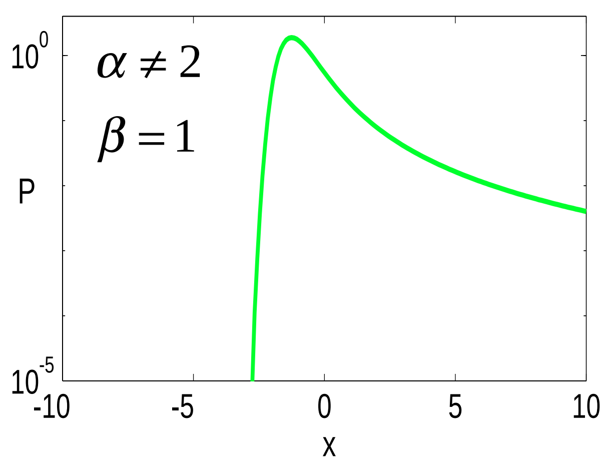
$$P(x,t) = t^{-\beta/\alpha} \, K(\eta) + O(\delta^2)$$

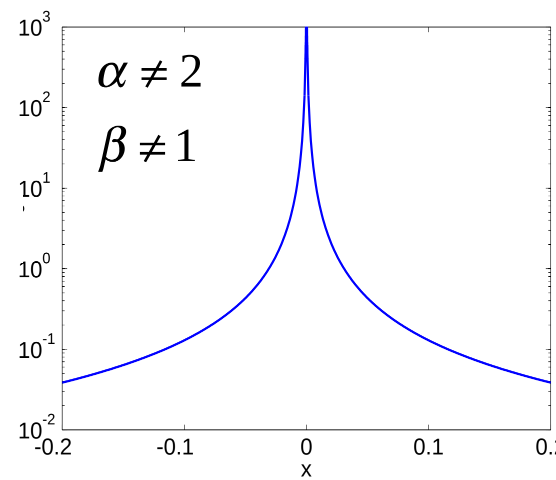
$$K(\eta) = \frac{1}{\pi} \int_0^{\infty} \cos(\eta z) \, E_{\beta}(z^{\alpha}) \, dz \qquad E_{\beta}(z) = \sum_n \frac{z^n}{\Gamma(\beta n + 1)}$$

Gaussian



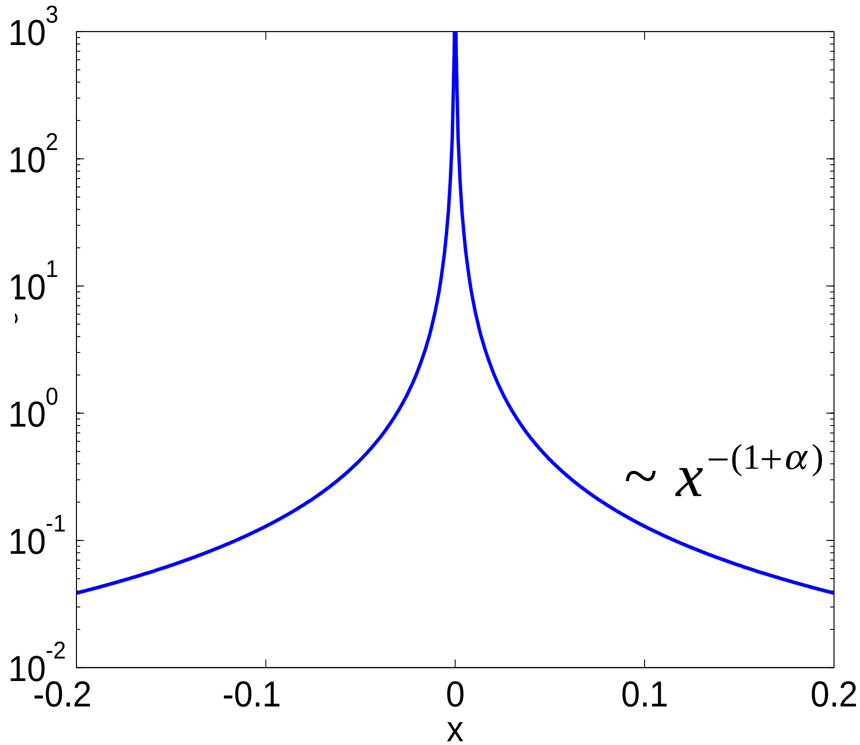
(asymmetric) Levy





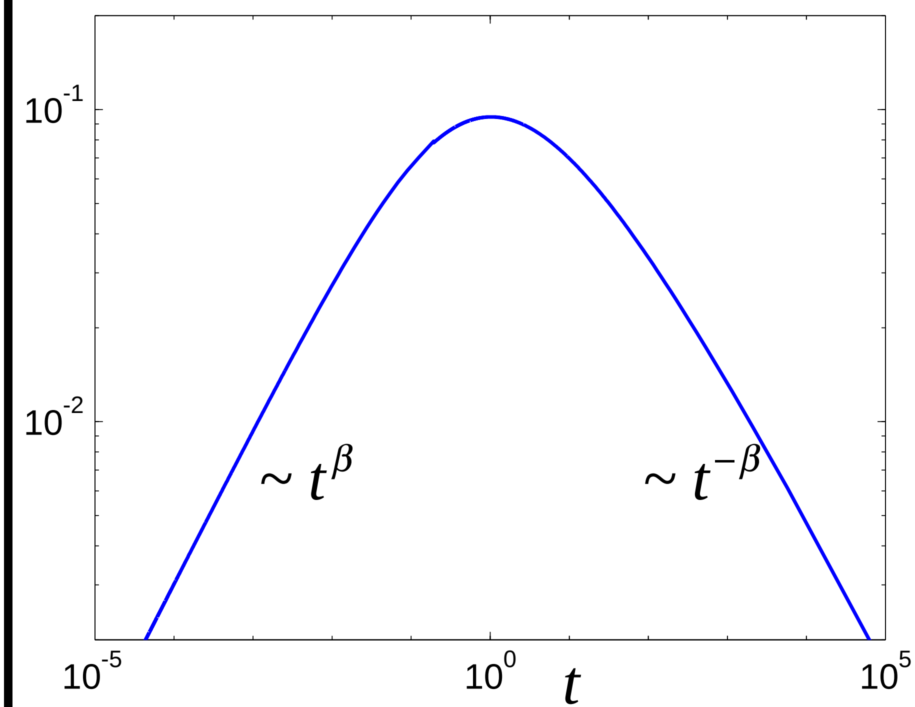
What is α?
space asymptotic scaling

$$P\left(x, t_0\right) \sim x^{-(1+\alpha)}$$



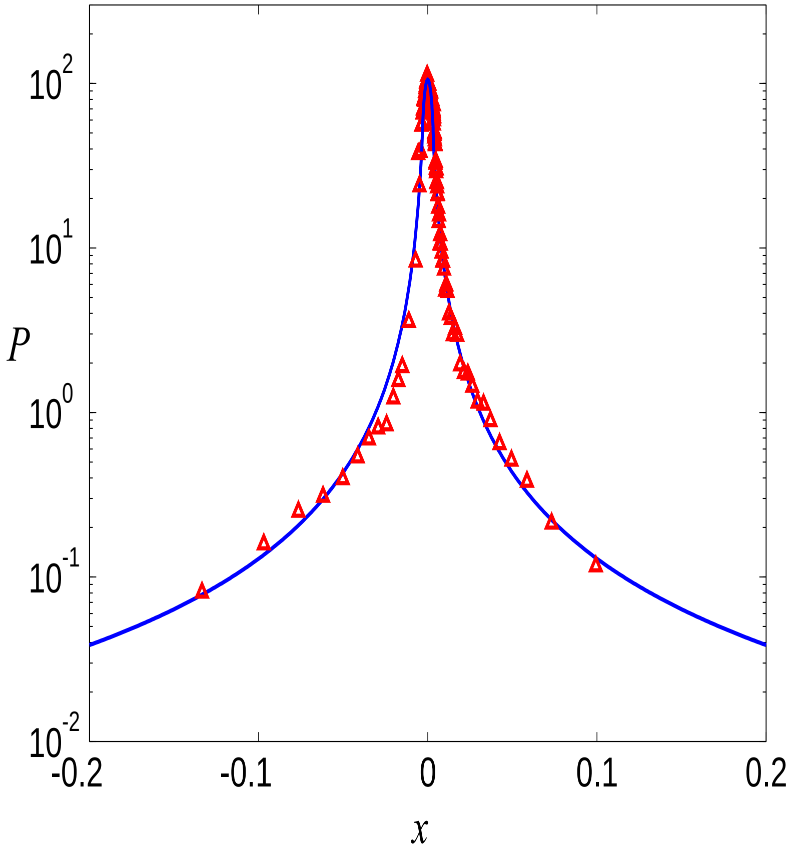
What is β?
time asymptotic scaling

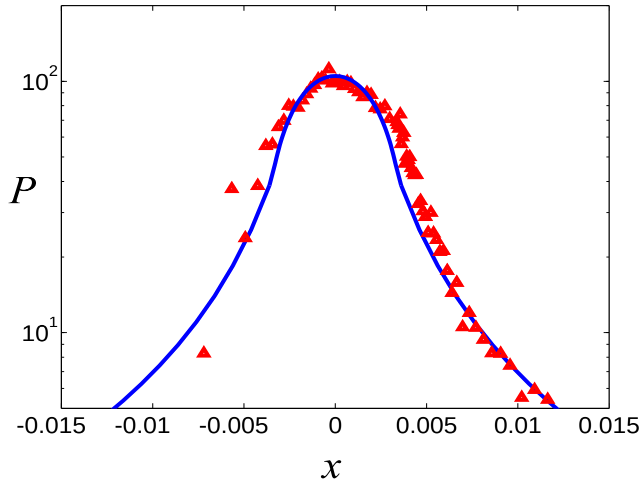
$$P\left(x_0, t\right) \sim \begin{cases} t^{\beta} \text{ for t} \sim 0 \\ t^{-\beta} \text{ for t} \to \infty \end{cases}$$

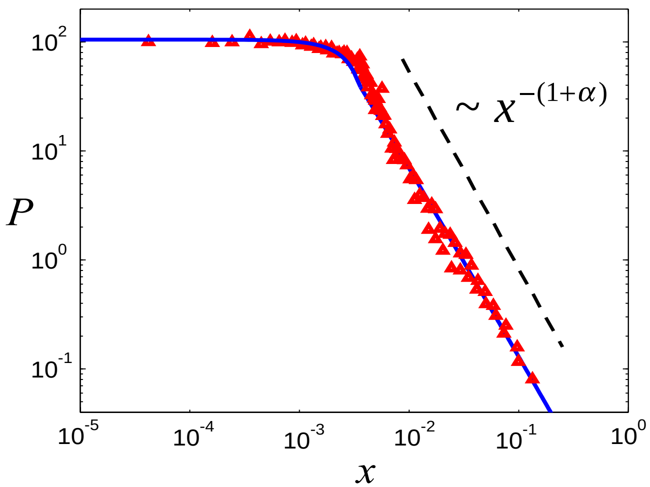


Comparison with fractional model
Probability density function of tracers

$\alpha=3/4, \quad \beta=1/2, \quad \chi=0.09$

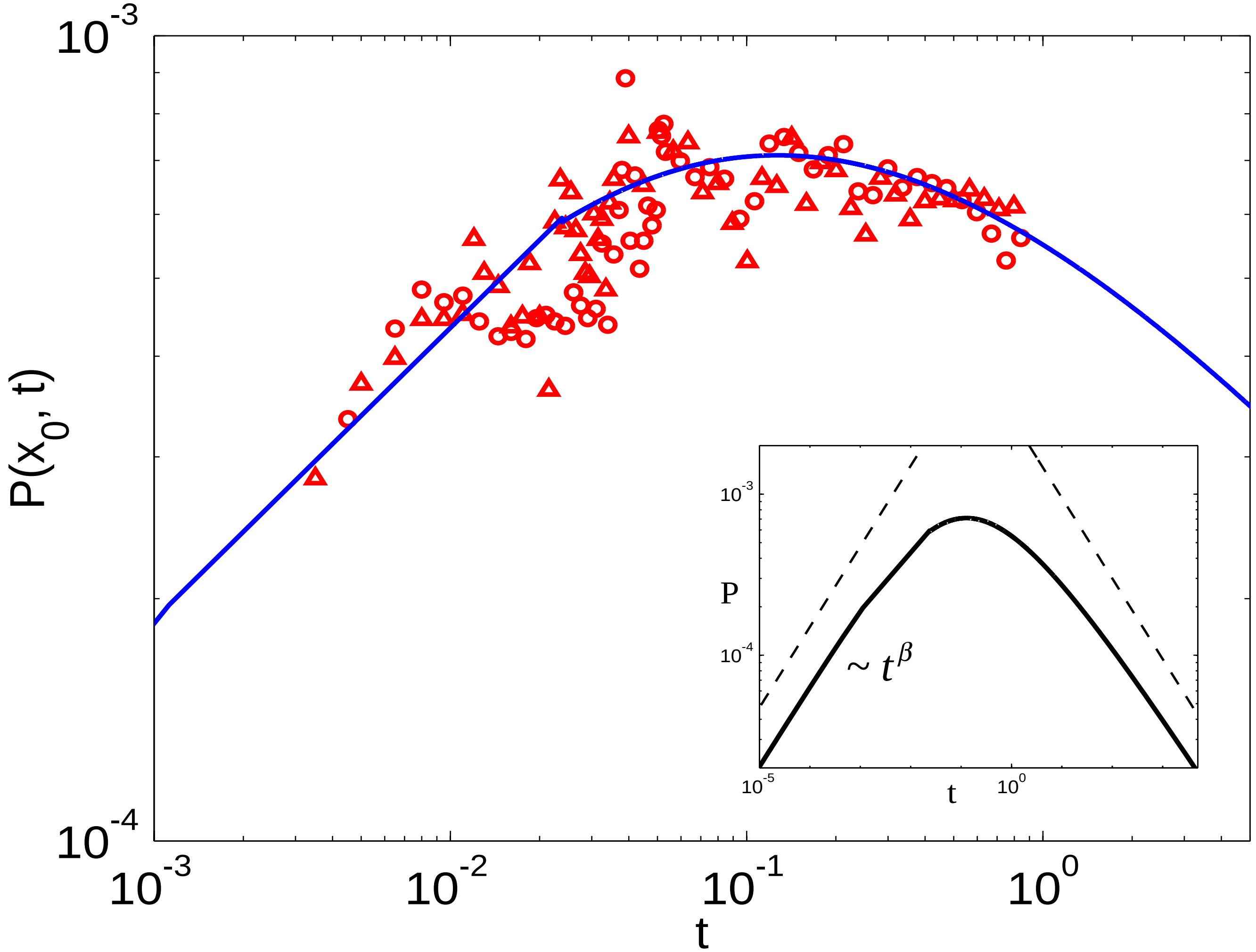






Comparison with fractional model
Probability density function of tracers

α=3/4, β=1/2, χ=0.09

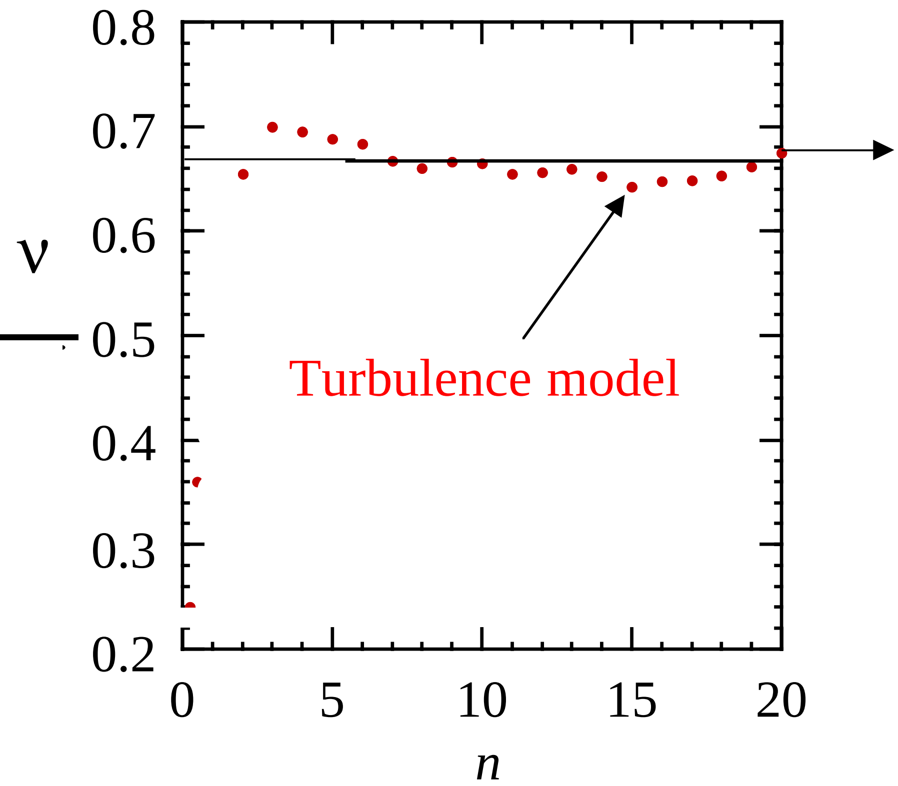


Scaling of moments

$$P(x,t) = t^{-\beta/\alpha} K(\eta) \qquad \eta = t^{-\beta/\alpha} x$$

$$\langle x^n \rangle = \int x^n P(x,t)\,dx = t^{n\beta/\alpha} \int \eta^n K(\eta)\,d\eta$$

$$\langle x^n \rangle \sim t^{n\nu}$$



Standard diffusion

$$\alpha = 3/4 \qquad \beta = 1/2$$

$$\nu = \beta/\alpha = 2/3$$

Fractional diffusion model

Summary

•We have shown numerical evidence of non-diffusive transport in plasma turbulence.

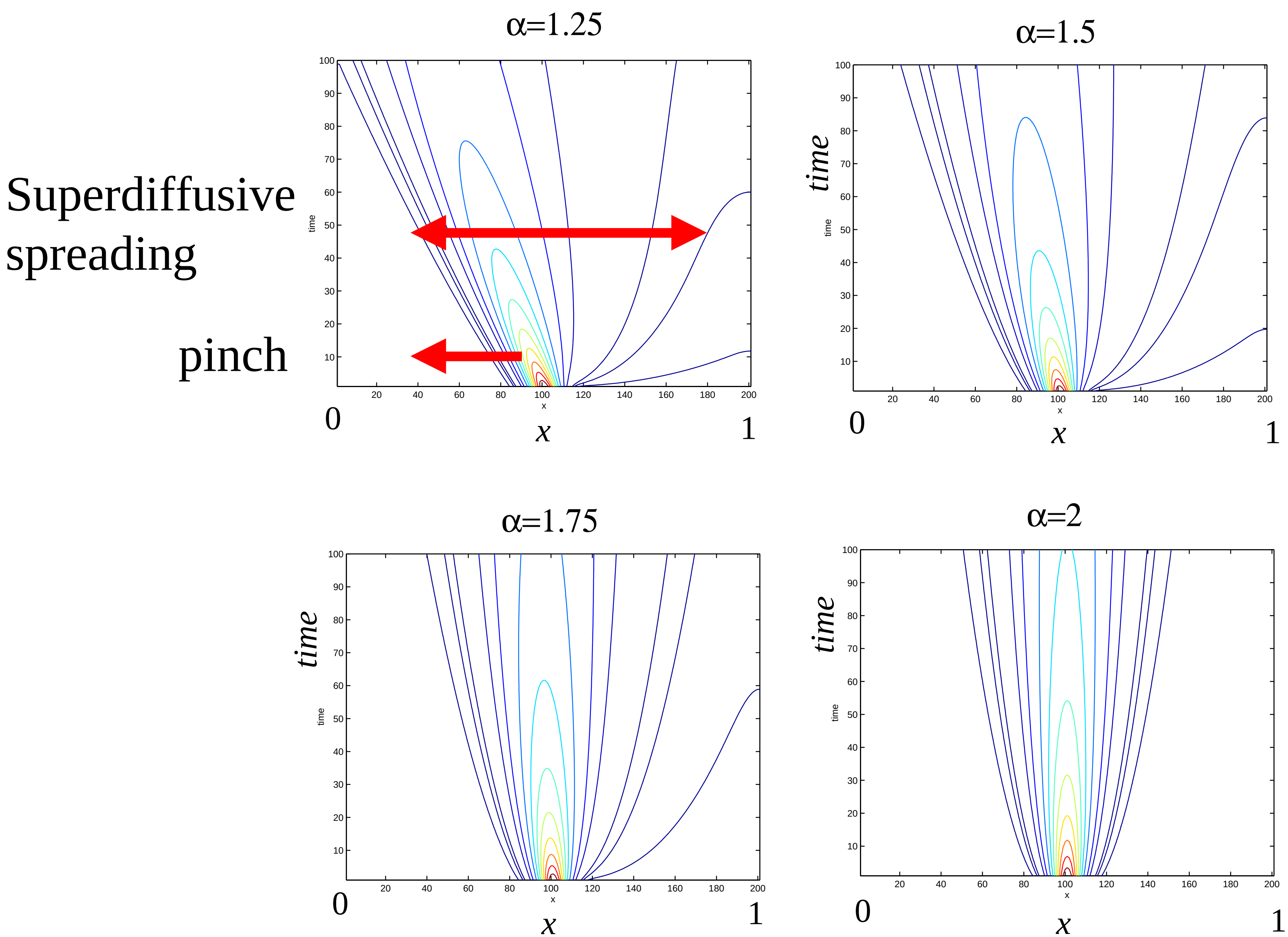
•We proposed a fractional transport model that incorporates in a unified way space non-locality (non-Fickian transport), memory effects (non-Markovian transport), and anomalous diffusion.

•There is quantitative agreement between the fractional model and the turbulence calculations.

Where do we go from here?

$$\partial_t^{\beta} P = c \underbrace{\left[{}^{-}\partial_x^{\alpha} + {}^{+}\partial_x^{\alpha} \right]}_{\text{Asymmetric fractional operators}} P + \underbrace{F(P)}_{\text{Nonlineratity}} + S$$

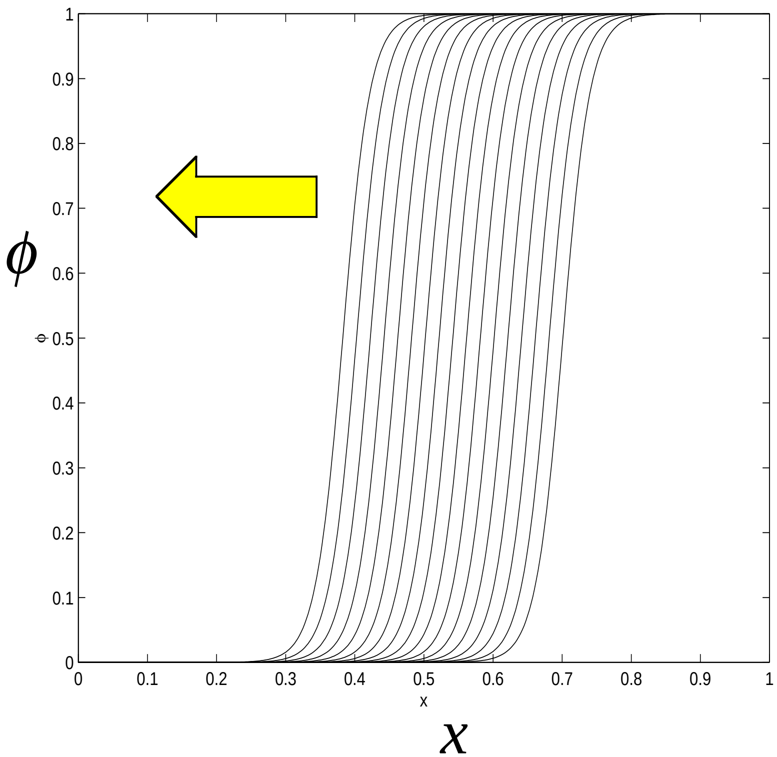
Asymmetric fractional transport of pulses



Fractional diffusion and nonlinearity

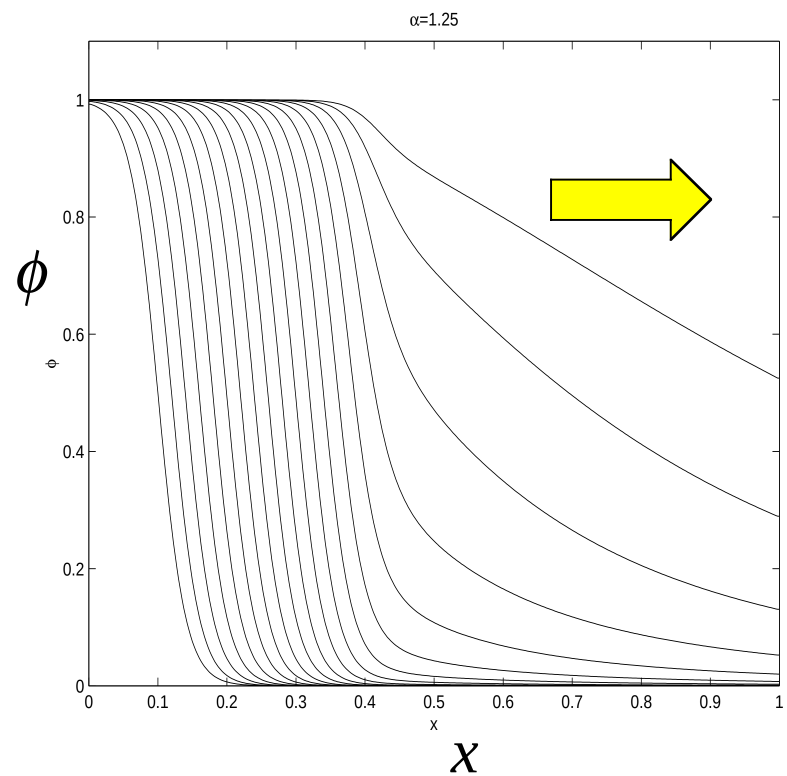
Exponential decaying
constant speed fronts

$\alpha = 2$



Algebraic decaying
accelerated fronts

$\alpha = 1.25$



What is a fractional derivative?

$$\frac{d^n f}{dx^n}$$

L'Hopital (1695):
"What if n=1/2?"

Leibniz (1695):
"This is an apparent paradox from which, one day, useful consequences will be drawn"

It is a usual practice to extend mathematical operations, originally defined for a set of objects, to a wider set of objects

$$\sqrt{r} \quad r \in \mathrm{R}^+ \quad \Rightarrow \quad \sqrt{s} \quad s \in \mathrm{R}$$

$$n! \quad n \in \mathrm{N}^+ \quad \Rightarrow \quad \Gamma(s) \quad s \in \mathrm{R}$$

$$\partial_x^n \phi \quad n \in \mathrm{N}^+ \quad \Rightarrow \quad \partial_x^\alpha \phi \quad \alpha \in \mathrm{C}$$

This mathematical "games" have, eventually, important physical applications

Fractional integral

Integer order integration

$$ {}_a\partial_x^{-n}\phi = \int_a^x dx_1 \int_a^{x_1} dx_2 \cdots \int_a^{x_{n-1}} dx_n\, \phi(x_n) $$

$$ {}_a\partial_x^{-n}\phi = \frac{1}{(n-1)!}\int_a^x (x-y)^{n-1}\,\phi(y)\,dy $$

Riemann-Liouville fractional integral

$$ \boxed{{}_a\partial_x^{-\nu}\phi = \frac{1}{\Gamma(\nu)}\int_a^x (x-y)^{\nu-1}\,\phi(y)\,dy} $$

Example: $$ {}_0\partial_x^{-\nu} x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}\, x^{\mu+\nu} $$

Fractional derivative

$$
{}_a\partial_x^{\mu}\,\phi = \frac{d^N}{dx^N}\left[{}_a\partial_x^{-\nu}\,\phi\right]
$$

$$
\nu = N - \mu
$$

$$
N = \text{smallest integer} > \mu
$$

$$
{}_a\partial_x^{m}\,\phi = \frac{d^N}{dx^N}\left[{}_a\partial_x^{-(N-m)}\,\phi\right] = \frac{d^m}{dx^m}\,\phi
$$

Riemann-Liouville fractional derivative

$$
{}_a\partial_x^{\alpha}\,\phi = \frac{1}{\Gamma(2-\alpha)}\,\partial_x^2 \int_a^x \frac{\phi\,(y)}{(x-y)^{\alpha-1}}\;dy
$$

$$
1 < \alpha \le 2
$$

Examples:

$$
{}_0\partial_x^{\mu}\,x^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}\,x^{\lambda-\mu}
\qquad
{}_{-\infty}\partial_x^{\mu}\,e^{ikx} = (ik)^{\mu}\,e^{ikx}
$$

Relation to asymmetric Levy distributions

$$\phi(x, t=0) = \delta(x)$$

$$\phi(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left[i^{\alpha} k^{\alpha} D t + i k x\right] dk$$

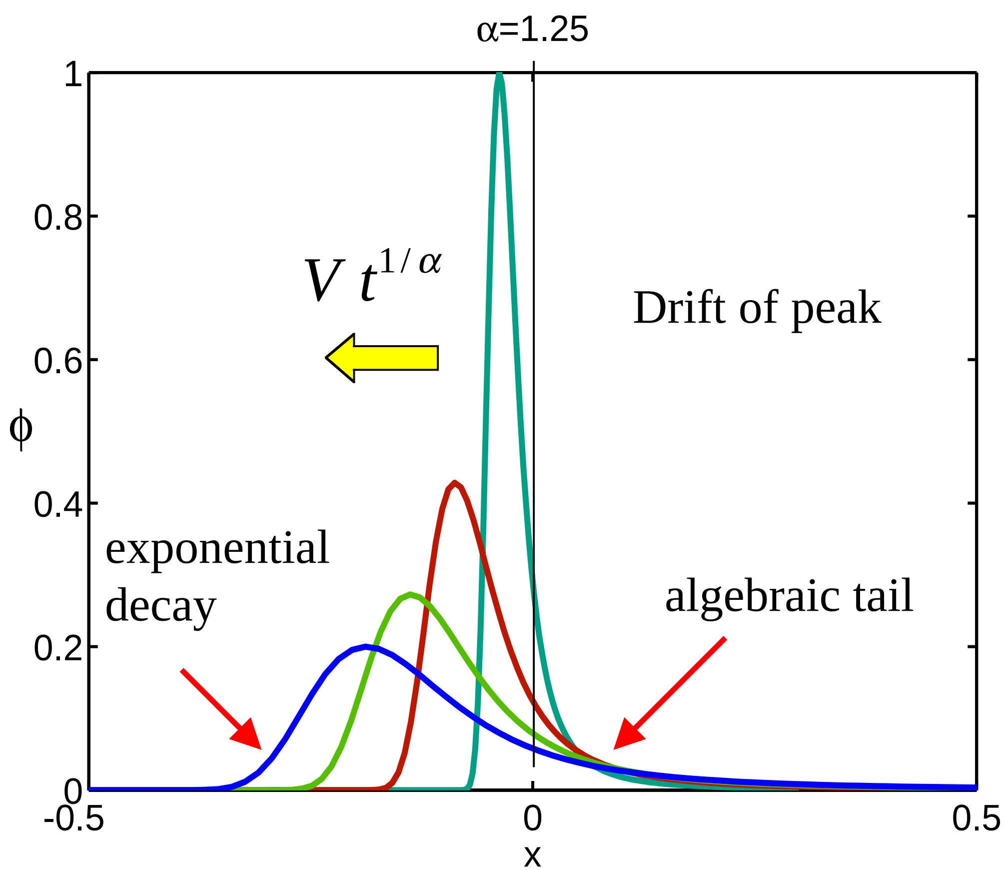
$$\tilde{\phi}(k,t) = \int_{-\infty}^{\infty} \phi(x,t)\, e^{-ikx}\, dx = e^{i^{\alpha} k^{\alpha} D t}$$

That is, the Fourier transform of the propagator of the fractional diffusion equation is the characteristic function of the extremal (asymmetric) Levy distribution $L_{\alpha,\beta}(x)$ with

$$0 < \alpha < 2 \qquad \beta = 2 - \alpha$$

Drift effect in fractional diffusion

Initial value problem $\phi(x,t=0)=\delta(x)$



$< x >= 0$

Sub-advection of peak concentration

$$x_*(t) = V(\alpha)\ t^{1/\alpha}$$

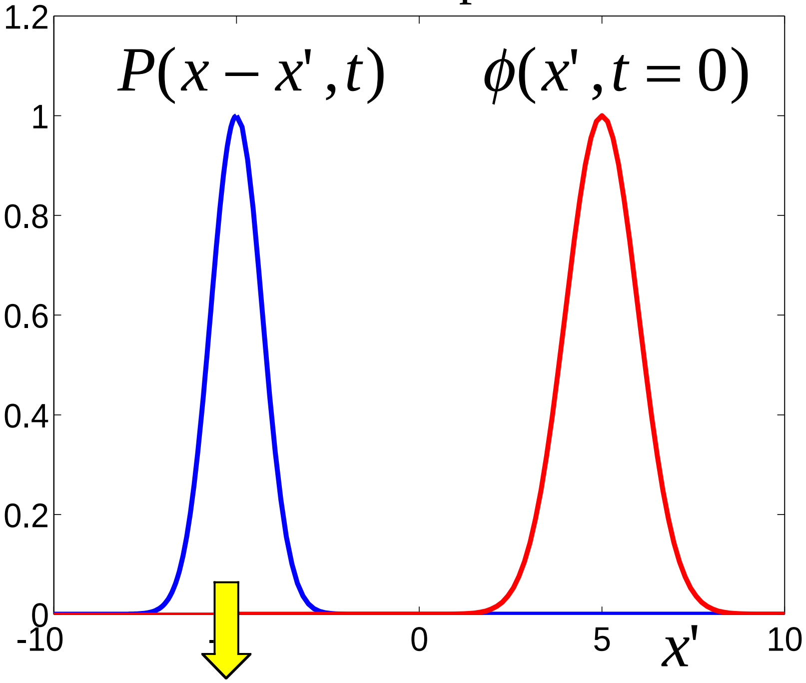
$$V = \left(\frac{\alpha+1}{2\alpha}\right)\tan\left(\frac{\alpha\pi}{2}\right)\left|\alpha\cos\left(\frac{\alpha\pi}{2}\right)\right|^{1/\alpha}$$

Nonlocal transport

$$\phi(x,t) = \int_{-\infty}^{\infty} dx'\ P(x - x';t)\,\phi(x',t=0)$$

$\lim_{x\to\infty} P \sim e^{-x^2}$

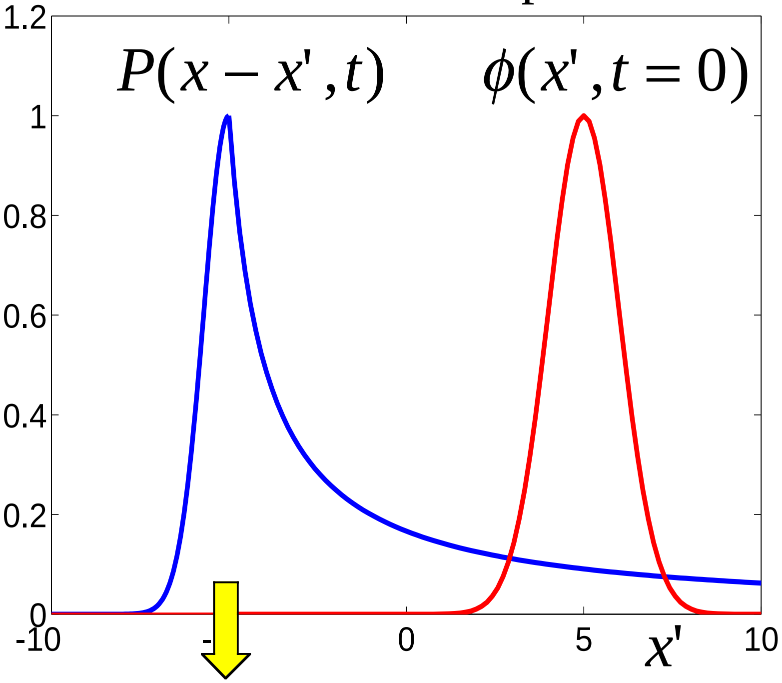
local transport



$\phi(x,t) \approx 0$

$\lim_{x\to\infty} P \sim x^{-(1+\alpha)}$

nonlocal transport



$\phi(x,t) \neq 0$

Due to the algebraic tail of the propagator, fractional diffusion leads to nonlocal transport

Nonlocal transport and global diffusive coupling

Grunwald-Letnikov definition $\partial_x^\alpha \phi = \lim_{h\to 0} \frac{\Delta_h^\alpha \phi}{h^\alpha}$

$$\Delta_h^\alpha \phi(x) = \sum_{j=0}^{N} (-1)^j \binom{\alpha}{j} \phi(x - j\,h)$$

$$\partial_x^\alpha \phi_k \approx \frac{\left(\Delta_h^\alpha \phi\right)_k}{h^\alpha} = \frac{1}{h^\alpha} \sum_{j=0}^{k} w_j^\alpha \, \phi_{k-j}$$

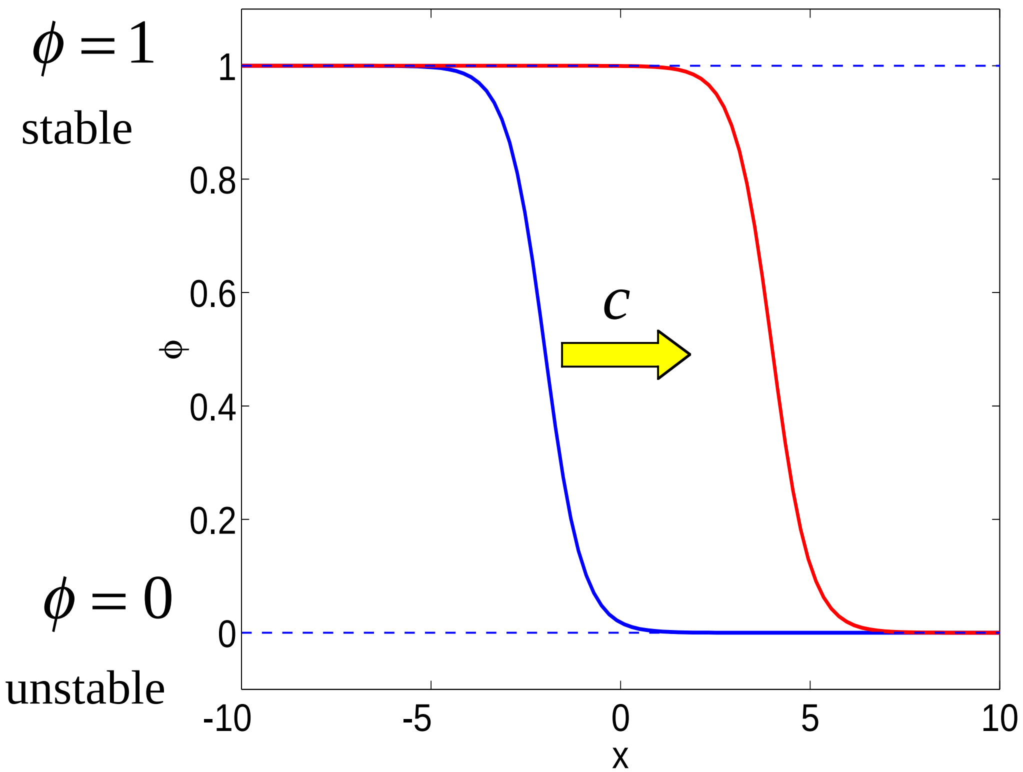
Finite difference approximation

$$w_0^\alpha = 1 \qquad w_j^\alpha = \left(1 - \frac{\alpha+1}{j}\right) w_{j-1}^\alpha$$

Global coupling coefficients

For α =1 and 2 usual nearest neighbor coupling of Laplacian operator
For 1 < α < 2 fractional diffusion implies global coupling

Reaction diffusion models of front dynamics



Fisher-Kolmogorov equation

$$\partial_t \phi = D\, \partial_x^2 \phi \;+\; \gamma\, \phi(1-\phi)$$

diffusion reaction

$$c = 2\sqrt{\gamma D}$$

dCN, te.al., Phys. Plasmas, **9**, 118, (2002), Physica **168D-169D**, 45, (2002).

- L-H transition models
- Model of genes dynamics
- Model of logistic population growth with dispersion
- Other applications include combustion and chemistry.

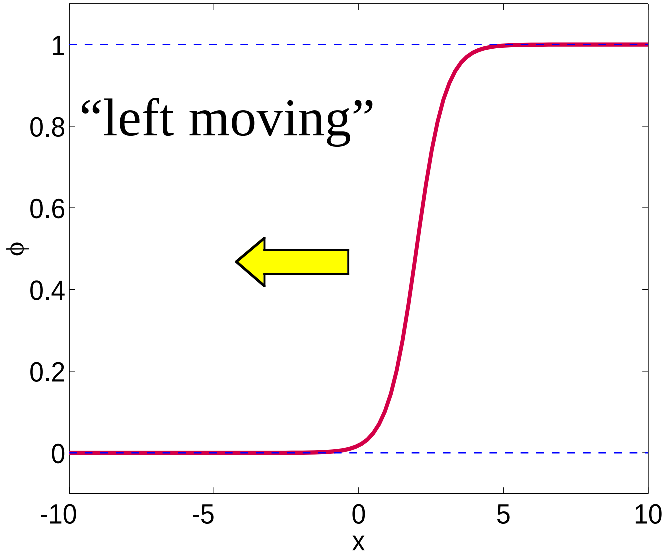
The fractional Fisher-Kolmogorov equation

$$\partial_t \phi = D \;{}_{-\infty}\partial_x^{\alpha} \phi + \gamma \, \phi \left(1-\phi\right) \qquad 1<\alpha<2$$

$${}_{-\infty}\partial_x^{\alpha} \phi = \frac{1}{\Gamma(2-\alpha)} \partial_x^2 \int_{-\infty}^{x} \frac{\phi\left(y\right)}{\left(x-y\right)^{\alpha-1}} \, dy$$

The asymmetry in the fractional derivative leads to an asymmetry in the fronts

Self-similar dynamics, exponential tails, constant front velocity



Quasi-self-similar dynamics, algebraic tails, front acceleration